CONCORDIA UNIVERSITY

Mathematical and Didactic Organization of Calculus Textbooks

An Extended Project in the Department of Mathematics and Statistics

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Abstract. *Mathematics textbooks are a great influence in the learning process at the university level as they are usually, together perhaps with the teacher, the students' main source of reference. Consequently, a study of Calculus textbooks can strongly contribute to our understanding on how students learn Calculus. This paper presents a study of the mathematical and didactic organizations of three selected Calculus textbooks. The goal is to bring to light the differences and similarities of those organizations to better understand the different teaching approaches to Calculus. The mathematical organization is analyzed from the perspective of the Anthropological Theory of the Didactic (ATD), in particular, considering the notion of "mathematical praxeology". The didactic organization is analyzed taking into account three different frameworks: ATD (in particular the notion of "didactic moments"), Model Readers, and formatting of students' behavior. The analysis of textbooks is focused on the mathematics topic algebra of derivatives and differentiation rules for two specific types of functions: the constant function and the power function. The results show important differences in both the mathematical and didactic organizations of the three textbooks. I discuss the possible implications of these differences for the learning of the concept of differentiation.*

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1. Introduction

Mathematics textbooks are a significant feature in the learning process of the students. They are often, together perhaps with the teacher, their main source of reference and therefore greatly affects how and what they learn. Consequently, to understand how students learn mathematics the study of mathematics textbooks is essential: "The book is still by far the most pervasive technology to be found in use in mathematics classrooms. Because it is ubiquitous, the textbook has profoundly shaped our notion of mathematics and how it might be taught. By its use of the 'explanation-example-exercises' format, by the way in which it address both teacher and learner, in its linear sequence, in its very conception of techniques, results and theorems, the textbook has dominated both the perceptions and the practices of school mathematics." (Love and Pimm, 1996) There is a great diversity of calculus textbooks used in our classrooms. Those textbooks, at the core, all roughly present the same theorems and definitions. These textbooks, however, contain numerous differences in both their mathematical and didactic approaches. Indeed, even though the topics are seemingly the same, the way the author chooses to present the topics will influence the learning process of the readers. "Researchers [...] have also suggested that differences in textbooks may affect student learning outcomes." (Weinberg and Wiesner, 2010, p.1)

This project is dedicated to the study of calculus textbooks. In particular, I looked into the mathematical and didactic discourses and analyzed the differences and similarities of three calculus textbooks. This project is not a critique of mathematics textbooks; in fact the goal is to highlight the different didactic and mathematical approaches to calculus from the point of view of the textbooks. Theorems, definitions, worked examples, etc. are all part of the mathematical layer. The didactic layer is everything that is meant to shape the readers interpretation of the text as well as shape how the reader uses the text. Both of those layers are analyzed for each of the textbooks studied and the results are compared and contrasted. The focus is on the mathematics topic algebra of derivatives and differentiation rules for two special functions – the constant function and the power function.

The analysis of the mathematical layer of the calculus textbooks is done here from the perspective of the Anthropological Theory of the Didactic (ATD; Chevallard, 1999, 2001). In this theory, Chevallard states that every human activity can be described by a unique model. This unique model is called "praxeology", which comes from the Greek praxis (practice) and logos (reason). As a result every regularly accomplished human activity, such as school mathematics activities, has a know-how block and a theoretical block. This model can be decomposed further into four elements: type of task, technique, technology and theory. Following this framework, I analyzed the mathematical praxeologies of the textbooks and I compared them against a reference praxeology, which is a theoretical model of the knowledge to be taught.

To analyze the didactic layer of the textbooks I considered the notion of *didactic moments* developed within the ATD. I have tried to locate the different didactic moments in each of the textbooks. These moments are part of the didactic process and for it to be successful, no matter how the process is established, those six moments have to take place at some point or another. These didactic moments are: *first encounter*, *exploration of the type of task and elaboration of a technique, construction of the theoretical block, work on the technique, institutionalization* and *evaluation*. Moreover, to further analyze the didactic layer, I analyzed the model readers of the textbooks as well as the notion of "formatting of the readers' behavior" (Sierpinska, 1997) with regards to the mathematical layer. To understand the model readers of the textbooks I used a framework from Weinberg and Wiesner (2010) who drew from reader-oriented theory; in particular, I considered the notions of *intended* reader and *implied* reader.

I chose three different textbooks on which to perform my analysis. The first textbook is "Single Variable Calculus Early Transcendentals" (Stewart, 2003). This textbook was the easiest choice due to it being commonly used in classrooms (at Concordia University and English Cegeps in Montreal). The second textbook also came naturally mainly because it is very different from what is usually seen in classrooms. It is Michael Spivak's "Calculus" (1980). Its rigorous approach to calculus sets it apart from the other two textbooks chosen and therefore brings a different point of view on the mathematical and didactic layers of calculus textbooks. Finally, the third book (Hughes-Hallett, Gleason et al., 1994) I chose is the written representative of the "Harvard Calculus Consortium principles" – these principles outline a teaching approach to calculus that was promoted in the late nineties, mostly at Harvard University (Knill, 2009). This book's approach differs from the approaches in the other two texts as it focuses on a geometrical interpretation of the mathematical topics.

This paper is organized in the following way. The next three chapters (Chapters 2 to 4) are dedicated to the analysis of the textbooks each based on a different framework: mathematical praxeologies, didactic moments and model readers. In Chapter 5 I present an analysis on how the textbooks' didactic layer may shape the readers' interpretation and utilization of the mathematical layer. Each of those four chapters contains an analysis for each textbook followed by a discussion section where the results are briefly examined. The last chapter, Chapter 6, presents the conclusions of these research and some ideas for future study. Finally, since Chapters 2 and 3 present only a sample of their respective analysis, Appendix A provides the full mathematical praxeology analysis while Appendix B presents the complete analysis of the didactic moments.

2. The Mathematical Layer in the Three Textbooks

2.1 Theoretical Framework and Reference Praxeology

As previously mentioned, the four elements of a praxeology – type of task, technique, technology and theory – characterize mathematical activity in an institution. The task refers to what must be accomplished and is usually defined by a verb. Every task t is categorized according to a type of task T; we write $t \epsilon T$. For example, to calculate the value of a function at a certain point is a type of task. However, *to calculate* is not a type of task, it is a task genre. A task genre is defined by the types of task associated with it and does not exist by itself. Furthermore, types of task and task genres do not occur naturally but are institutionally built. The main purpose of the didactics is therefore to reconstruct the tasks in an institution, in this specific case in the mathematical textbooks.

The task is the first component of the know-how block of the praxeology. The second component is the technique (τ); together, the type of task and the technique represent the *savoir-faire* or know-how block $[T/\tau]$. A technique is a method available to accomplish the tasks $t \epsilon T$. A technique, however, might work only for some particular cases of the tasks $t \epsilon T$. In other words, the technique has a specific reach beyond which it fails. For instance, every technique fails to factor natural numbers when they get too large. Indeed, cryptography uses the fact that one cannot *in general* factor natural numbers. An institution may have only one technique or a small group of techniques for a certain type of task. Other institutions may have different techniques.

The knowledge block is also formed by two components which are the technology and the theory. The technology θ is the rational discourse about the technique. Its purpose is to justify the use of the technique τ for a certain type of task T. The technology ensures that the technique used will in fact accomplish the tasks of the type T and explains why it does so. The technology has a third function which is the production of new techniques. Thus there exists what can be called *potential* technologies which are technologies to no or very few techniques. Some technologies are therefore under exploited in regard to both justification and production.

The theory Θ is the last component of the praxeology and also the second component in the knowledge block $[\theta/\Theta]$. The technological discourse, like the technique, contains some assertions that require explanations. The theory is consequently another level of justification, explanation and production. Just as the technology provides the reasoning behind the technique, the theory is the rationale behind the technology.

To illustrate what a praxeology is let's take an example from the field of mathematics. Consider the type of task T: to find how many measurements lie within k standard deviations of the mean for a given mound shaped distribution. The technique τ is to apply the empirical rule which states that 68% of the measurements lie within 1 standard deviation of the mean, 95% of the measurements lie within 2 standard deviations of the mean and 99.7% of the measurements lie within 3 standard deviations of the mean. To justify the use of this technique, we have to look at the technology θ. Since the distribution is mound shaped it can be approximated by the normal probability distribution. The area under the normal curve between a and b represents the probability of a measurement y falling in the interval$[a, b]$. Using the statistic table for the standardized normal distribution, the area under the curve between -1 and 1 is $0.6826 \cong 68\%$, between -2 and 2 is $0.9544 \cong 95\%$ and finally between -3 and 3 is $0.9974 \cong 99.7\%$. The corresponding theory Θ can be extensive. It first involves proving that any mound shaped distribution can be approximated using the normal distribution. Then, different theoretical aspects related to the normal probability distribution are necessary. Some of these aspects are the density function of the normal probability distribution $f(y) = (\frac{1}{\sigma\sqrt{2\pi}})e^{-(y-\mu)^2/(2\sigma^2)}$ and calculating the area under that curve by integrating $f(y)$ for the given values of a and b with $\sigma = 1$ and $\mu = 0$.

2.1.1 Punctual, Local and Regional Praxeology

There are at least three levels into which we can analyze a mathematical praxeology. The most basic level is the punctual praxeology which is the praxeology that is developed around a unique type of problem. The second level is the local praxeology. A local praxeology integrates various punctual praxeologies that use the same technological discourse. The final level is the regional praxeology. A regional praxeology combines different local praxeologies that share the same theoretical discourse. An example of a punctual praxeology could be centered on solving a system of two linear equations with two unknowns. A local praxeology encompassing this punctual praxeology could be to solve n linear equations with m unknowns. Finally, the regional praxeology would be built around the theory of vector spaces.

2.1.2 Reference Praxeology

A reference mathematical praxeology is "our epistemological model of the 'scholarly knowledge' that legitimates the knowledge to be taught. It is the broader map with reference to which we can interpret the mathematical contents that are proposed to be studied at school." (Barbé et al., 2005, p. 241)

To analyze and discuss the mathematical knowledge presented in the textbooks, I will compare the textbooks' praxeologies against a reference praxeology. In the analysis, I focused on the algebra of derivative and on the differentiation rules for two specific types of functions: the constant function and the power function. There are other functions that have specific rules of differentiation such as the exponential functions. However, since I won't address those in my analysis I built the reference praxeology only for those two previously identified functions. The corresponding reference mathematical praxeology that I take into account can be globally described by a regional praxeology that I will call "the derivative of functions".

Regional Reference Praxeology: "the derivative of functions"

Type of Task: T Given $f(x)$ find $f'(x)$.

Technique: If possible, use τ_1, τ_2 or a combination of them. If this is not possible, use τ_3.

τ_1 Apply the algebra of derivatives

τ_2 Apply the rules for special functions

τ_3 Find the derivative by definition

Technology:

θ_1

θ_{11}: $f(x) = cg(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{cg(x+h)-cg(x)}{h}$$

$= \lim_{h\to 0} c\left[\frac{g(x+h)-g(x)}{h}\right]$

$= c\ \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$

$= cg'(x)$

θ_{12}: $f(x) = g(x) + h(x)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{[g(x+h)+h(x-h)]-[g(x)+h(x)]}{h}$

$= \lim_{h\to 0} \left[\frac{g(x+h)-g(x)}{h} + \frac{h(x+h)-h(x)}{h}\right]$

$= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} \frac{h(x+h)-h(x)}{h}$

$= g'(x) + h'(x)$

If $f(x) = g(x) - h(x)$

Rewrite $f(x)$ as $g(x) + (-1)h(x)$. Using technique τ_1 for the derivative of a constant times a function and the technique τ_2 regarding the derivative of the sum of two functions we get

$f'(x) = g'(x) + (-1)h'(x) = g'(x) - h'(x)$

θ_{13}: $f(x) = g(x)h(x)$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{(gh)(x+h)-(gh)(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)h(x+h)-g(x)h(x)}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)h(x+h)-g(x+h)h(x)+g(x+h)h(x)-g(x)h(x)}{h}$$

$$= \lim_{h\to 0} \left[\frac{g(x+h)[h(x+h)-h(x)]}{h} + \frac{[g(x+h)-g(x)]h(x)}{h}\right]$$

$$= \lim_{h\to 0} g(x+h) \lim_{h\to 0} \frac{h(x+h)-h(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \ \lim_{h\to 0} h(x)$$

$$= g(x)h'(x) + g'(x)h(x)$$

θ_{14}: $f(x) = \ g(x)/h(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\left(\frac{g}{h}\right)(x+h)-\left(\frac{g}{h}\right)(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{g(x+h)}{h(x+h)}-\frac{g(x)}{h(x)}}{h}$$

$$= \lim_{h\to 0} \frac{g(x+h)h(x)-g(x)h(x+h)}{h[h(x)h(x+h)]}$$

$$= \lim_{h\to 0} \frac{g(x+h)h(x)-g(x)h(x)+g(x)h(x)-g(x)h(x+h)}{h[h(x)h(x+h)]}$$

$$= \lim_{h\to 0} \frac{[g(x+h)-g(x)]h(x)-g(x)[h(x+h)-h(x)]}{h[h(x)h(x+h)]}$$

$$= \lim_{h\to 0} \left(\frac{[g(x+h)-g(x)]h(x)}{h} - \frac{g(x)[h(x+h)-h(x)]}{h}\right)\frac{1}{h(x)h(x+h)}$$

$$= \left[h(x) \lim_{h\to 0} \frac{[g(x+h)-g(x)]}{h} \ - g(x) \lim_{h\to 0} \frac{h(x+h)-h(x)}{h}\right] \lim_{h\to 0} \frac{1}{h(x)h(x+h)}$$

$$= \frac{[h(x)g'(x)-g(x)h'(x)]}{[h(x)]^2}$$

θ_2

θ_{21}: If $f(x) = c$ then $f'(x) = 0$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} =$$

$$= \lim_{h\to 0}\frac{c-c}{h} = 0$$

θ_2: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left[x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n\right]-x^n}{h}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n}{h}$$

$$= \lim_{h\to 0}\left[nx^{n-1}+\frac{n(n-1)}{2}x^{n-2}h+\cdots+nxh^{n-2}+h^{n-1}\right]$$

$$= nx^{n-1}$$

θ_{22} : If $f(x) = x^n$ then $f'(x) = nx^{n-1}$

$$f(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{(x+h)^n-x^n}{h}$$

$$= \lim_{h\to 0}\frac{\left[x^n+nx^{n-1}h+\frac{n(n-1)}{2}x^{n-2}h^2+\cdots+nxh^{n-1}+h^n\right]-x^n}{h}$$

$$= \lim_{h\to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$$= nx^{n-1}$$

θ_3 $\quad f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$

Theory Θ

$\varepsilon - \delta$ definition of limits.

Algebra of limits (limits laws) along with their proofs.

This regional praxeology integrates (a) a local praxeology that I will call "the algebra of derivative" and (b) a local praxeology that I will call "derivative of special functions". The algebra of derivative focuses on the addition, subtraction, multiplication and division of functions and also on the multiplication of constant times a function. The derivative of special functions focuses on those functions that have special differentiation rules. As mentioned before, I only looked at two types of functions: the constant function and the power function. Other functions with special differentiation rules would also fit into this local praxeology but are not presented. There may also be other local praxeologies in this regional praxeology. However, since I chose to analyze only these two local praxeologies to get an understanding of how the textbooks present these mathematical topics, I will only need the reference mathematical praxeology for these specific local praxeologies.

Reference local mathematical praxeology for the algebra of derivatives

Type of Task T_1: Find $f'(x)$ when $f(x)$ is obtained by performing algebraic operations on functions (multiplication of a function by a constant, or addition, subtraction, product or quotient of two functions).

Technique: τ_1 Apply the algebra of derivatives

If $f(x) = cg(x)$ then use τ_{11}: $f'(x) = cg'(x)$

If $f(x) = g(x) \pm h(x)$ then use τ_{12}: $f'(x) = g'(x) \pm h'(x)$

If $f(x) = g(x)h(x)$ then use τ_{13}: $(f)'(x) = g'(x)h(x) + g(x)h'(x)$

If $f(x) = g(x)/h(x)$ then use τ_{14}: $f'(x) = \frac{g'(x)h(x) - g(x)h'(x)}{[h(x)]^2}$

Technology: The techniques are justified by the definition of derivatives. More specifically:

θ_1 (see above $\theta_{11}, \theta_{12}, \theta_{13}, and\ \theta_{14}$)

Theory Θ

An example of a punctual praxeology that is integrated in the above described local praxeology could be:

Type of Task : Find $f'(x)$, where $f(x) = cg(x)$

Technique : $f'(x) = cg'(x)$

Technology: $f(x) = cg(x)$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{cg(x+h) - cg(x)}{h}$$

$$= \lim_{h\to 0} c\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= c\ \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= cg'(x)$$

Theory Θ

Reference local praxeology for the derivative of the special functions constant function and power function

Type of Task: T_{21} Find $f'(x)$ if $f(x) = c$.

T_{22} Find $f'(x)$ if $f(x) = x^n$.

Technique : τ_{21} $f'(x) = 0$.

τ_{22} $f'(x) = nx^{n-1}$

Technology θ: Those two techniques are justified using the definition of derivative of functions.

θ_2 (See θ_{21} and θ_{22})*Theory Θ*

The local praxeology "the algebra of derivative" integrates four punctual praxeologies, which will be called: "derivative of a constant times a function", "derivative of addition/subtraction of functions", "derivative of a product of functions" and "derivative of a quotient of functions". The second local praxeology is limited to two special functions which are the constant function and the power function therefore it has two punctual praxeologies: "constant function" and "power function".

2.2 Mathematical Praxeologies in the three textbooks

In this section, I analyze the praxeologies in each of the selected textbooks, related to the mathematics topics mentioned before. A selection of the textbooks' praxeologies is presented here while all the praxeologies analyzed are presented in full in Appendix A. In all the analysis, textbooks are coded Books A, B and C, corresponding to Stewart (2003), Hughes-Hallett et al. (1994) and Spivak (1980), respectively.

According to the reference praxeologies defined above, the six types of tasks to be analyzed are the following:

Type of Task 1: Differentiate the function $f(x) = c$, where c is a constant.

Type of Task 2: Differentiate $f(x) = x^n, n \in \mathbb{R}, n \neq 0$.

Type of Task 3: Differentiate $f(x) = cg(x)$ where c is a constant.

Type of Task 4: Differentiate $f(x) + g(x)$ and $f(x) - g(x)$.

Type of Task 5: Differentiate $f(x)g(x)$

Type of Task 6: Differentiate $f(x)/g(x)$

Those six types of tasks are presented in different orders in the three books. The order for each book is the following:

Stewart (Book A)	Hughes-Hallett, Gleason, et al. (Book B)	Spivak (Book C)
Type of task 1	Type of task 1	Type of task 1
Type of task 2	Type of task 3	Type of task 4
Type of task 3	Type of task 4	Type of task 5
Type of task 4	Type of task 2	Type of task 3
Type of task 5	Type of task 5	Type of task 2
Type of task 6	Type of task 6	Type of task 6

Table 1. Order of types of tasks in each of the three textbooks.

2.2.1 Book A

Type of Task 1: Differentiate the function $f(x) = c$, where c is a constant.

Technique τ_1: $f'(x) = 0$.

Technology θ_1

The technique is justified by stating that since the graph of the function is a horizontal line with slope 0, $f'(x) = 0$.

A formal proof of the technique is provided involving the definition of derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h}$$

Theory Θ_1

$\varepsilon - \delta$ definition of limits.

Algebra of limits (laws of limits).

Type of Task 2: Differentiate $f(x) = x^n, n \in \mathbb{R}, n \neq 0$.

Technique τ_2: $f'(x) = nx^{n-1}$

Technology θ_2

Two justifications of the technique τ_2 are provided for positive integer values of n. Both use the definition of derivative. The first justification starts by giving the equation $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$ which we are told can be verified by multiplying out the right hand side or by summing the second factor as a geometric series. The rest of the proof is the following:

If $f(x) = x^n$ then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x^n - a^n}{x - a}$$

$$= \lim_{x \to a} (x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$$

$$= (a^{n-1} + a^{n-2}a + \cdots + aa^{n-2} + a^{n-1})$$

$$= na^{n-1}$$

A second justification of the technique τ_2 is provided also using the definition of derivative. The difference is that the Binomial Theorem is used to expand $(x + h)^n$ (a side note tells us where we can find this theorem in the appendix).

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h}$$

$$= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h}$$

$$= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$= nx^{n-1}$ because every term except the first has *h* as a factor and therefore approaches 0.

The author mentions that the rule is true for every real value of n and directs the students to the section about the derivative of logarithmic functions for the proof:

Let $y = x^n$

$\ln|y| = \ln|x|^n = n \ln|x|$ where $x \neq 0$. Therefore

$$\frac{y'}{y} = \frac{n}{x}$$

$$y' = n\frac{y}{x} = n\frac{x^n}{x} = nx^{n-1}$$

Theory Θ_2

$\varepsilon - \delta$ definition of limits

Algebra of limits (Limit Laws)

Definition of derivatives

Binomial Theorem

Type of Task 5: Differentiate $f(x)g(x)$

Technique τ_5: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Technology θ_5

The justification of the technique is geometrical but still uses the notion of derivative as the basis of the proof.

The proof starts by assuming that $u = f(x)$ and $v = g(x)$ are both positive differentiable functions. Then, the product uv is interpreted as the area of a rectangle. Hence, if x changes by an amount Δx, then the corresponding changes in u and v are $\Delta u = f(x + \Delta x) - f(x)$ and $\Delta v = g(x + \Delta x) - g(x)$. The new value of the product $(u + \Delta u)(v + \Delta v)$ can be interpreted as the area of the larger rectangle.

The change in the area of the rectangle is

$$\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u\Delta v$$

Divide by Δx:

$$\frac{\Delta(uv)}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}$$

If we let $\Delta x \to 0$, we get the derivative of uv:

$$\frac{d}{dx}(uv) = \lim_{\Delta x \to 0}\frac{\Delta(uv)}{\Delta x} = \lim_{\Delta x \to 0}\left(u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}\right)$$

$$= u\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x} + v\lim_{\Delta x \to 0}\frac{\Delta u}{\Delta x} + \left(\lim_{\Delta x \to 0}\Delta u\right)\left(\lim_{\Delta x \to 0}\frac{\Delta v}{\Delta x}\right)$$

$$= u\frac{dv}{dx} + v\frac{du}{dx} + 0\frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

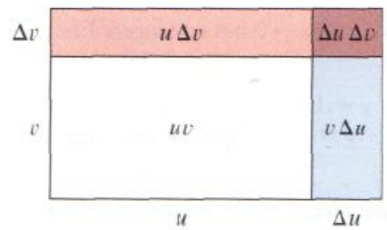

FIGURE 1
The geometry of the Product Rule

(Stewart, 2003, p. 192)

There is a note mentioning that even though the justification starts with the assumption that all the quantities are positive to allow for the geometrical interpretation, the equation $\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u\Delta v$ is always valid.

Theory Θ_5

$\varepsilon - \delta$ definition of limits

Algebra of limits (Limit Laws)

Definition of derivatives

Discussion

Comparing this praxeology to the reference praxeology, the first thing that comes to mind is the lack of regional and local praxeologies. In Book A, these all seem to be standalone tasks with the unifying goal of avoiding using the definition. As stated in the introduction to Chapter 3 Differentiation Rules, "it would be tedious if we always had to use the definition, so in this chapter we develop rules for finding derivatives without having to use the definition directly." (Stewart, 2003, p. 183) As a result what we have is a list of punctual praxeologies without it never being mentioned that they all are part of the same regional praxeology. However, one part that is exactly the same in Book A as in the reference praxeology is the theory. Technologies are also very similar except in a few cases discussed next.

Book A's technologies are very definition oriented. However, in the justification of the product rule θ_5 it may be more difficult for the students to see how the definition of the derivative is involved. Yet, a side note remind the student that $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$. This technology is quite different from the reference praxeology but still uses the definition although in a very different way. Furthermore, Book A occasionally offers two different technologies for the same technique providing different point of views. For the constant function, the author justifies the derivative being 0 by the slope of the tangent line to the graph which is a horizontal line everywhere. Then the author provides a formal proof using the definition. Doing so, the author reminds the reader of the geometrical interpretation of the derivative as well as its formal interpretation. Also, the proof for the derivative of the power function is done in two ways. However, none is given more importance than the other but both bring something different to the understanding of the topic. The first proof starts with $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \lim_{x \to a} \frac{x^n-a^n}{x-a}$ and then factors out $(x-a)$ from the numerator while the other starts with $f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n-x^n}{h}$ and factors the numerator using the Binomial theorem. This last proof introduces the students to the Binomial Theorem and is a complete and sufficient justification of the technique. The first proof reminds the students of equivalent statements of the definition of the derivative.

The justifications of the proofs are mostly devoid of comments and don't make reference to the theory except when limit laws are used. In such cases, a note in parenthesis will tell the reader the number of the law that was used. These numbers refer to a numbering that was done when those laws were first introduced in a previous chapter. If more than one law is used the author chooses to only alert the reader that a given step was accomplished using limit laws. This is to say that theory is referenced only in those cases and in the recurrent usage of the definition of derivative.

2.2.2 Book B

Type of Task 1: Differentiate the function $f(x) = c$, where c is a constant.

Technique τ_1: $f'(x) = 0$.

Technology θ_1

The technique is justified by the fact that the graph of a constant function is a horizontal line with a slope of 0 everywhere therefore the derivative is 0.

Theory Θ_1

Definition of derivatives

Informal definition of limits: We define $\lim_{h\to c} g(h) = L$ to mean that $g(h)$ can be made as close to the number L as we like by choosing h close enough to c. (Hughes-Hallett, Gleason, et al., 1994, p. 138)

Type of Task 2: Differentiate $f(x) = x^n, n \in \mathbb{R}, n \neq 0$.

Technique τ_2: $f'(x) = nx^{n-1}$

Technology θ_2

$$\frac{d}{dx}(x^n) = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h\to 0}\frac{[x^n + nx^{n-1}h + \cdots + h^n] - x^n}{h}$$

$$= \lim_{h\to 0}\frac{nx^{n-1}h + \cdots + h^n}{h}$$

$$= \lim_{h\to 0}\frac{h(nx^{n-1} + \cdots + h^{n-1})}{h}$$

When you factor out h from terms involving h^2 and higher powers of h, each term will still have an h in it. Performing the division we get:

$$\frac{d}{dx}(x^n) = \lim_{h\to 0}(nx^{n-1} + \cdots + h^{n-1})$$

But as $h \to 0$, all terms involving an h will go to 0, so

$$\frac{d}{dx}(x^n) = \lim_{h\to 0}(nx^{n-1} + \cdots + h^{n-1}) = nx^{n-1}$$

For negative and fractional powers of n, the book states that those derivatives can be calculated using the definition. As an example, x^{-2} is differentiated and the authors state that the rule holds for negative and fractional powers even though they are not in a position to prove this yet.

Further investigation in the book shows that no formal justification for negative and fractional powers is offered. However, in a section about applications of the chain rule, they use the chain rule to differentiate $x^{1/n}$ thus showing that the rule applies to fractional powers. The example is the following (Hughes-Hallett, 1994, p.226):

Differentiate $x^{1/n}$

$$\left(x^{\frac{1}{n}}\right)^n = x$$

Differentiating both sides gives

$$n\left(x^{\frac{1}{n}}\right)^{n-1} \frac{d}{dx}\left(x^{\frac{1}{n}}\right) = 1$$

$$\frac{d}{dx}\left(x^{\frac{1}{n}}\right) = \frac{1}{nx^{(n-1)/n}} = \frac{1}{nx^{1-\left(\frac{1}{n}\right)}} = \frac{1}{n}x^{\left(\frac{1}{n}\right)-1}$$

Theory Θ_2

Definition of derivatives

Informal definition of limits: We define $\lim_{h\to c} g(h) = L$ to mean that $g(h)$ can be made as close to the number L as we like by choosing h close enough to c.

Type of Task 6: Differentiate $f(x)/g(x)$

Technique τ_6:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Technology θ_6:

The technology for the technique τ_6 uses the technique to find the derivative of a product of functions τ_5:

Let $Q(x) = \frac{f(x)}{g(x)}$. Since $f(x) = Q(x)g(x)$, we can use the product rule,

$$f'(x) = Q'(x)g(x) + Q(x)g'(x) = Q'(x)g(x) + \frac{f(x)}{g(x)}g'(x)$$

Solving for $Q'(x)$: $$Q'(x) = \frac{f'(x) - \frac{f(x)}{g(x)}g'(x)}{g(x)}$$

Multiplying top and bottom by $g(x)$ to simplify gives

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{\left(g(x)\right)^2}$$

Theory Θ_6:

Informal definition of limits: We define $\lim_{h \to c} g(h) = L$ to mean that $g(h)$ can be made as close to the number L as we like by choosing h close enough to c.

A praxeology corresponding to the task: differentiate the product of two functions.

Discussion

The main difference with the reference praxeology is the lack of cohesiveness in the presentation of the techniques. There is no regional and no local praxeology. What is presented is a catalogue of punctual praxeologies. For each specific type of task, there is a different punctual praxeology to choose from.

The theory in this book is not as rigorous as the theory of the reference praxeology. For instance, the limit laws are nowhere presented in the book though they are used a few times to justify the techniques. Following the justification of the technique for the differentiation of a sum of functions, the authors wrote: "You may have noticed that we have used the fact that the limit of a sum is the sum of the limits – a fact which is true, although we have not proved it. (Can you see why it is true?)" (Hughes-Hallett, Gleason, et al., 1994, p.189) This question is never answered in the book and it's hard to imagine what the authors expect the reader to answer. Also, the $\varepsilon - \delta$ definition of limits is not presented in the textbook. The only definition is an informal one which states: "We define $\lim_{h \to c} g(h) = L$ to mean that $g(h)$ can be made as close to the number L as we like by choosing h close enough to c." (Hughes-Hallett, Gleason, et al., 1994, p. 138) Therefore, if the theory of Book B is compared to the reference praxeology theory, only the definition of derivative is present in Book B, thereby missing the limit laws along with their proofs and the $\varepsilon - \delta$ definition of limits.

Book B provides a technology for every technique; for some, however, the focus is less on the definition of derivative and more on a graphical justification. For instance, the justification of the technique to differentiate a constant function states that since the graph of the function is a horizontal line, the slope of the tangent line is 0. This technology is the same as in Book A. These authors, however, don't provide another justification using the definition. This is again in contrast with the reference praxeology. The other proofs of techniques all involve using the definition except for the quotient of functions where the product rule is used and is yet another difference with the reference praxeology. Another important point to make is the fact that no justification is provided for the technique to differentiate a difference of functions. In addition to this missing technology, the technology for the differentiation of a power function $f(x) = x^n$ is limited to positive exponents. However, as mentioned above, an example introduced later in the book exhibits weak traces of the missing technology: the example verifies the technique for the derivatives of power functions with fractional powers. Also worth mentioning is the fact that the Binomial Theorem is used in the proof of the power rule for positive exponent but is neither referenced nor explained. The theory is also not referenced in any of the justifications of the techniques except when the technologies use the definition of derivative.

2.2.3 Book C

Type of Task 2: Differentiate $f(x) = x^n, n \in \mathbb{R}, n \neq 0$.

Technique τ_2: $f'(x) = nx^{n-1}$

Technology θ_2

The justification of the technique τ_2 is by induction. For $n = 1$ the book refers to a theorem previously proven using the definition of derivative that showed that if $f(x) = x$, then $f'(a) = 1$ for all a:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \to 0} \frac{a + h - a}{h} = \lim_{h \to 0} \frac{h}{h} = 1$$

Assume it is true for n, so that if $f(x) = x^n$, then $f'(a) = na^{n-1}$ for all a. Let $g(x) = x^{n+1}$. If $I(x) = x$, the equation $x^{n+1} = x^n x$ can be written $g(x) = f(x)I(x)$ for all x.

The book then uses technique τ_5 to find the derivative of a product of functions which has been previously provided in this book to find the derivative of $g(a)$:

$g'(a) = (fI)'(a) = f'(a)I(a) + f(a)I'(a)$

$= na^{n-1}a + a^n 1$

$= na^n + a^n$

$= (n+1)a^n$, for all a

This is precisely the case $n+1$ which we wished to prove.

Theory Θ_2

$\varepsilon - \delta$ definition of limits

Algebra of limits (limit laws)

A praxeology associated to the task: differentiate the product of two functions

Type of Task 5: Differentiate $f(x)g(x)$

Technique τ_5: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$

Book C also gives the technique to differentiate $(fgh)(x)$

$(fgh)'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$

Technology θ_5

$$(fg)'(a) = \lim_{h \to 0} \frac{(fg)(a+h) - (fg)(a)}{h}$$

$$= \lim_{h \to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(a+h)[g(a+h)-g(a)]}{h} + \frac{[f(a+h)-f(a)]g(a)}{h}\right]$$

$$= \lim_{h\to 0} f(a+h) \lim_{h\to 0}\frac{g(a+h)-g(a)}{h} + \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \lim_{h\to 0} g(a)$$

$$= f(a)g'(a) + f'(a)g(a)$$

Theory Θ_5

$\varepsilon - \delta$ definition of limits

Algebra of limits (laws of limits)

Type of Task 6: Differentiate $f(x)/g(x)$

Technique τ_6:

$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

Technology θ_6:

Book C splits the justification of the technique τ_6 into two theorems. The first theorem shows that $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{[g(a)]^2}$. Before providing the proof of this theorem, the author mentions that it is necessary to make sure that $\left(\frac{1}{g}\right)(a+h)$ is defined for small h:

Since g is, by hypothesis, differentiable at a, then g is continuous at a (proven in a previous theorem). Also, since $g(a) \neq 0$, there is some $\delta > 0$ such that $g(a+h) \neq 0$ for $|h| < \delta$. Therefore, $\left(\frac{1}{g}\right)(a+h)$ does makes sense for small enough h, and we can write

$$\lim_{h\to 0} \frac{\left(\frac{1}{g}\right)(a+h) - \left(\frac{1}{g}\right)(a)}{h} = \lim_{h\to 0} \frac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h}$$

$$= \lim_{h\to 0} \frac{g(a) - g(a+h)}{h[g(a)g(a+h)]}$$

$$= \lim_{h\to 0} \frac{-[g(a+h) - g(a)]}{h} \frac{1}{g(a)g(a+h)}$$

$$= \lim_{h\to 0} \frac{-[g(a+h) - g(a)]}{h} \lim_{h\to 0} \frac{1}{g(a)g(a+h)}$$

$$= -g'(a) \frac{1}{[g(a)]^2}$$

After this theorem, Book C introduces the technique τ_6 and provides its justification using the technique to find the derivative of a product of functions τ_5.

Since $\frac{f}{g} = f\left(\frac{1}{g}\right)$ we have

$$\left(\frac{f}{g}\right)'(a) = \left(f\ \frac{1}{g}\right)'(a)$$

$$= f'(a)\left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$$

$$= \frac{f'(a)}{g(a)} + \frac{f(a)(-g'(a))}{[g(a)]^2}$$

$$= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$$

Theory Θ_6:

$\varepsilon - \delta$ definition of limits

Algebra of limits (laws of limits)

A praxeology associated to the task: differentiate the product of two functions

Discussion

The introduction to chapter 10 called "Differentiation" provides a certain connection between the definition of derivative and the differentiation techniques using the different techniques presented above. Talking about differentiation the author says: "From the previous chapter you may have the impression that this process is usually laborious, requires recourse to the definition of the derivative, and depends upon successfully recognizing some limit. It is true that such a procedure is often the only possible approach – if you forget the definition of the derivative you are likely to be lost. Nevertheless, in this chapter we will learn to differentiate a large number of functions, without the necessity of even recalling the definition." (Spivak, 1980, p. 154) This can be seen as the techniques to a regional praxeology where the type of task is to find the derivative of a function and the techniques are τ_1: use the definition and τ_2: use special rules. The theory in Book C is also the same as the theory in the reference praxeology.

Similar to the reference praxeology, Book C uses the definition in some way or another to justify every technique except to justify the technique to differentiate a constant times a function. The main difference with the reference praxeology is that some tasks are split into two parts giving interesting technologies. For instance, the technique to differentiate $f(x) = x$ is introduced as a standalone technique. Then, to justify the technique to differentiate a power function $f(x) = x^n$, this result is recalled and then the rest of the proof is by induction on n. The proof by induction can be difficult to grasp however it is a very effective tool. Since the proof that the technique holds for $n = 1$ is done using the definition, this justification is based on the definition of derivative. Also, the technique to differentiate the product of functions is used to justify the technique for a constant times a function and for a quotient of functions. In the latter case, this is combined with a previously proved technique to differentiate $f(x) = \left(\frac{1}{g(x)}\right)$ which was justified using the definition. The author makes almost no reference to the theory in his proofs. However, in two instances he did justify one step of a proof. In his proof of the technique for the derivative of a product of functions and for the derivative of $f(x) = \left(\frac{1}{g(x)}\right)$, he justifies writing $\lim_{h\to 0} f(a+h) = f(a)$ (or $\lim_{h\to 0} g(a+h) = g(a)$) by referring the reader to a previous theorem stating that if f is differentiable at a, then f is continuous at a. The other proofs have no reference to the theory other than using the definition of derivative.

2.3 Discussion

The three textbooks offer different praxeologies therefore presenting the mathematical topics in their own unique ways. The techniques and types of tasks are similar. However, the differences occur mainly in the way the techniques are presented, in the technologies and in the theories.

The reference praxeology is cohesive in the sense that there is a regional praxeology and local praxeologies to tie all the punctual praxeologies together. In contrast, Book A and Book B don't provide any regional praxeology to help the student navigate between this list of techniques and the definition of derivative. Those two books give the impression that those rules will replace the technique of differentiating using the definition and that this definition can safely be forgotten. On the other hand, Book C does mention that in some cases the definition will be the only approach available and that forgetting it would be a mistake. He therefore presents the techniques as an alternate way to differentiate functions though in some cases, the students will have to return to the definition. As opposed to the other two books, Book C has a regional praxeology.

The technologies in the books are in general quite similar to the reference praxeology. The authors mostly use the definition to justify the techniques. The book that presents the most differences with the reference praxeology and with the other two books is Book B. This book is missing some technologies as in the case where the technique for the difference of functions is not justified and the power function is justified only for positive powers and for fractional powers. The other two books present technologies for every technique using the definition or a technique previously justified by the definition. This is another difference with Book B that justifies the technique to differentiate a constant function only graphically with no reference to the definition.

Finally, another important point to make is in regard to the theory. The reference praxeology presented a theory that is matched by Book A and Book C. However, Book B doesn't present the same theory. In fact, some theory is completely missing from the textbook. The $\varepsilon - \delta$ definition of limit is nowhere to be found in the book; only an informal definition is given in terms of limits. Also, the limit laws are missing. Those are not listed nor explained in the book however they are used in a few technologies. This is a major difference from the reference praxeology and the other books that present a complete theory sufficient to justify the technology used.

3. The Didactic Layer in the Three Textbooks

3.1 Theoretical Framework

According to the Anthropological Theory of the Didactic, (ATD, Chevallard, 1999), a didactic process has a structure that can be characterized by six *moments*. Each of these *moments* has "a specific function to fulfill which is essential for the successful completion of the didactic process" (Barbé et al., 2005, p.238). No matter how a (successful) learning path is organized, there will be *moments* where certain learning motions will have to be accomplished. The six moments may vary qualitatively and quantitatively and may not appear in any given order. However all six of them have to appear sometime during the didactic process for it to be successful.

The first moment is the *first encounter* with the mathematical praxeology that will be studied. One can encounter the object under study in different ways but unless one wants to only graze the surface of the topic, the first encounter inexorably happens through a type of task that is part of the praxeology that is being studied. This first encounter could also take place many times; one can rediscover a type of task that was previously considered known. In its most rigid version, the first encounter would also include the object's *raison d'être*, the motivation behind its creation.

The second moment is the moment of the *exploration* of the type of task T and the *elaboration of a technique* τ relative to T. Solving a problem of a type of task T always involves generating at least the beginning of a technique. From this embryo of a technique, a more elaborated technique can eventually emerge. Consequently, the study of a particular problem of certain type of task T is a tool to create a technique τ that will subsequently be used almost routinely to solve problems of the same type.

The third didactic moment is the *construction of the theoretical block* $[\theta/\Theta]$ relative to τ. This moment is interrelated with every other moment. Indeed, right from the first encounter with a certain type of task, a connection is usually established with previously studied knowledge blocks. Traditionally, according to Chevallard (1999), this moment is the first step in the learning process and is the same for many types of task T, which are every types of task that share the same knowledge block $[\theta/\Theta]$. Each of the problems from these types of task will then be applications of this particular knowledge block.

The fourth moment is the *work on the technique*. This moment is to improve the technique to make it more effective and more reliable. This is when the technique is tested usually through problems and exercises, which entails retouching the technique if necessary.

The fifth moment is the *institutionalization*. This is when the mathematical praxeology is clearly defined. This moment states what elements used in the construction of the praxeology will actually be integrated in it and which elements won't be. This is the answer to the question students frequently ask and that all teachers are well accustomed with: Do we need to know this? The first four moments listed above are part of the construction of the praxeology. This one defines exactly what this praxeology is (according to the particular institution that is defining it).

The sixth and last moment is the *evaluation*. The evaluation is constructed around the institutionalization. At this moment, the value of what has been learned is analyzed based on the definition of the praxeology that has been established at the moment of institutionalization. In addition to evaluating the individual, the evaluation moment should also be used to look at the technique itself to determine its strength and usability. The evaluation is formative for the praxeology and can showcase the need to revisit certain moments.

3.2 Didactic Moments in the Three Textbooks

As previously mentioned, the six didactic moments have to appear throughout the learning process for it to be successful. I analyzed the three books and located, when they were present, the six didactic moments for each of the six praxeologies previously presented. However, I only present some of those results here; full description of all identified didactic moments can be found in Appendix B.

3.2.1 Book A

Moments #3 and #5 are the same for any of the praxeologies described below. Moment #6 doesn't occur at all.

Moment #3: Construction of the theoretical block $[\theta, \Theta]$ relative to τ

The construction of the knowledge block was done in previous sections of the book.

Definition of Derivative:

The definition of derivative was given as well as its interpretations as rate of change, velocity and slope of a tangent line. Furthermore, a section was dedicated to the recognition of the derivative as a function.

Limits:

Limits are introduced with the following definition: We write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L if we can make the values of $f(x)$arbirtrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a." Then, the Limit Laws are presented without proofs. Finally, the $\varepsilon - \delta$ definition of limits is given.

The definition of derivative and of limit is frequently used in the technology to justify the techniques presented. The limit laws are also used in the technology and they are referenced by their numbers, which refer to the numbering done when those laws were first introduced.

Moment #5: Institutionalization

Institutionalization starts at the beginning of the chapter. The introduction to chapter 3 mentions that the point of this chapter is to develop rules to easily find the derivatives of various functions without having to use the definition of derivative, since using the definition can be tedious. Such functions include polynomial, rational and algebraic functions. The book also points out that those rules will be used to solve word problems involving rate of change and approximation of functions. Also, the examples and exercises provided in the textbook – by focusing on differentiating functions using the rules as well as focusing on the interpretations of the derivative as the slope of a tangent line and as a rate of change – are implicitly part of the institutionalization. They let the readers know what part of the mathematical layer they should concentrate on. The readers are therefore aware of what this chapter attempts to achieve and what information they need to retain.

<u>Praxeology "Constant Function"</u>

Moment #1: First encounter

The introduction to Chapter 3 called "Differentiation Rules" states that rules to differentiate functions are going to be introduced. It also says that "in this chapter we develop rules for finding derivatives without having to use the definition directly." (Stewart, 2003, p. 183) This provides the motivation behind the praxeology.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

The elaboration of the technique focuses on the graph of the function. Since the graph of the function $f(x) = c$ is a horizontal line with slope 0 then the derivative of $f(x)$ must be 0.

Moment #4: Work on the technique

This moment doesn't appear following the presentation of the technique. However, this technique will be used late in the book in combination with other techniques to work on different types of task as well as in exercises.

<u>Praxeology "Power Function"</u>

Moment #1: First encounter

The type of task of this praxeology has been encountered in a preceding chapter as first an example to find a tangent line to a graph and then as a way to better understand how to use the definition of derivatives.

The introduction to Chapter 3 called "Differentiation Rules", as previously mentioned, provides the motivation behind the praxeology.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

The elaboration of the technique starts with an example where the derivative of $f(x) = x$ is found. Book A explains that the derivative is 1 since the graph of the function is $y = x$ which has slope 1 everywhere. Following this example, the reader is reminded of an exercise of the previous chapter where the derivatives of $f(x) = x^2$ and $g(x) = x^3$ were found by approximation using a graphing device and then by using the definition of derivatives. Lastly, the derivative of $f(x) = x^4$ is found using the definition of derivative and the reader's attention is drawn to the emergent pattern before finally stating the technique.

Moment #4: Work on the technique

The technique is first presented to be only for positive integer values of n. Then, some examples are provided with the first example being solved by applying the technique directly and involving only n's that are positive integers.

Then another example is provided involving a negative exponent. This example is solved using the technique. However, in the exercises at the end of the section the students are asked to prove this specific result using the definition and they are also asked to show that the technique holds for negative exponents in general.

An example using a rational exponent ($\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$) is also solved using the technique. This example was solved in a previous section where the slope of the tangent line to this function was found using the definition.

The book then states that the proof that this technique is true for all real values of n is provided in a future section using logarithmic differentiation but the rule is still generalized at this point to include all real values of n. Finally, this technique has to be used in the exercises section where the worked examples may serve as templates to guide the readers.

Praxeology "Derivative of a Product of Functions"

Moment #1: First encounter

The introduction to Chapter 3 called "Differentiation Rules", as previously mentioned, provides the motivation behind the praxeology.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

The reader discovers the technique to differentiate a product of functions geometrically, beginning with a rectangle and then changing the lengths of its side therefore changing the area.

The book starts by assuming that $u = f(x)$ and $v = g(x)$ are positive differentiable functions. The product uv can be interpreted as the area of a rectangle. If x changes by an amount Δx, then the corresponding changes in u and v are $\Delta u = f(x + \Delta x) - f(x)$ and $\Delta v = g(x + \Delta x) - g(x)$ and the value of the product $(u + \Delta u)(v + \Delta v)$ can be interpreted as the area of the larger triangle.

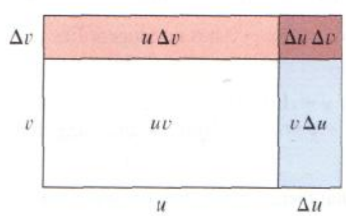

FIGURE 1
The geometry of the Product Rule

(Stewart, 2003, p. 194)

The change in the area of the rectangle is $\Delta(uv) = (u + \Delta u)(v + \Delta v) - uv = u\Delta v + v\Delta u + \Delta u \Delta v$.

After some manipulations and taking limits, the following rule to differentiate a product of functions appears:

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Moment #4: Work on the technique

Four examples are provided to work on the technique. The first example requires differentiating $f(x) = xe^x$ using the technique. The technique to differentiate exponential functions has been provided previously in the book. The second example also asks to differentiate a function using the technique. However, this example also provides an alternate solution using the laws of exponents and the techniques for the differentiation of the power function and of the sum of functions. The third example is again a computational example where the derivative is found using the technique. Finally, the fourth example is a word problem where the function is not given in the problem and has to be set up. The question asks to find the rate of increase. The function is set up by multiplying two other functions and the derivative is found using the technique to find the rate of increase. Lastly, this technique has to be used in the exercises section of the chapter where the worked examples can serve as models to help the readers solve the problems.

Discussion

The first encounter is present at least partially for all the praxeologies. The introduction to the chapter "Differentiation" provides the motivation behind the praxeologies. Moreover, some types of task were previously encountered in the preceding sections as worked examples about rate of change or to illustrate how to use the definition of derivative.

Moment #2, the exploration of the types of task and the elaboration of a technique is lacking for some praxeologies in Book A. Indeed, the techniques about the multiplication of a function by a constant and about the addition and subtraction of functions are stated without elaboration. On the other hand, other techniques are developed both geometrically and using the definition. For instance, before presenting the technique to differentiate a power function, this type of task is explored geometrically for $f(x) = x$ and using the definition for $f(x) = x^4$. Then the book invites the student to see the emergent pattern before finally stating the technique. Furthermore, the techniques for the product and the quotient of functions are explored graphically and elaborated using the definition and rate of change. Finally, the technique to find the derivative of a constant function is developed graphically using the tangent line to a graph. As a result, Book A integrates various interpretations of derivative in this moment.

The third moment, when the theoretical block is built, is in fact the first moment to appear in the book. The theory is constructed in previous sections such as the definition of derivative on which the technologies are based.

The fourth moment is present in every praxeology. Mostly this moment is seen in the worked examples following the presentation of the technique. Those examples are all solved using the technique and none require using the definition. An interesting aspect of this moment is that some examples are word examples asking to find rates of change or equations of tangent lines to a graph. This reminds the student of the different interpretations of the derivative. This moment also appears in the exercises sections of the textbook; in particular, for the praxeology “constant function”, it appears only in the exercises. Finally, in the praxeology “power function” and “derivative of addition/subtraction of functions”, this moment is used to extend the techniques. The “power function” praxeology initially presented the technique only for positive integer exponents. Then, examples involving negative and fractional exponents brought about the extension of the technique to include all real exponents. The same thing happened for the “derivative of addition/subtraction of functions” praxeology where the technique for the addition of functions was used to develop the technique for the subtraction of function.

The fifth moment is the same for every praxeology and takes place in the introduction of the chapter “Differentiation”. The readers know from reading the introduction that the praxeologies of this chapter are about learning rules to easily differentiate various functions without having to use the definition of derivative. They also know that they will have to use those techniques to solve problems involving rates of change.

Moment #6 doesn’t appear in Book A for any praxeology.

3.2.2 Book B

Moment #3: Construction of the theoretical block $[\theta, \Theta]$ relative to τ.

Definition of Derivative:

The theoretical block is constructed in previous sections of the book. The definition of derivative is provided along with the interpretations of the derivative as a rate of change and as the slope of a tangent line to the graph of a function. The derivative is also looked at as velocity and the second derivative as acceleration. Moreover, there are sections on how to interpret the first and second derivative with regards to the graph of the function (increasing, decreasing or constant and concave up or concave down).

Limits:

Limits are introduced in a section called "Notes on the Limit". The $\varepsilon - \delta$ definition of limits is not introduced, an informal interpretation of the limit, however, is provided: "We define $\lim_{h \to c} g(h) = L$ to mean that $g(h)$ can be made as close to the number L as we like by choosing h close enough to c" (Hughes-Hallett, Gleason, et al, 1994, p.138). The limit laws are also not provided but they are used in the justifications of techniques.

The definition of derivative and the interpretation of the derivative as the slope of a tangent line to a graph are used in the technology of the techniques. Limits are also used along with the limit laws. However, those laws are not part of the theoretical block. Book B doesn't reference the theoretical block but it uses it.

Moment #5: Institutionalization

The institutionalization starts in the introduction to chapter 4 called "Short-Cuts to Differentiation". The introduction states that this chapter is a systematic study of the derivatives of functions given by formulas and that general rules such as the product and the quotient rules will be provided to allow the differentiation of combinations of functions. Reading this, the students are then aware of what the praxeologies will be about and what they have to know by the end of the chapter. Also, the examples and exercises focus on computations and on the graphical interpretation of the derivative. They are therefore part of the institutionalization process, implicitly telling the readers what part of the mathematical layer they should retain.

Moment #6: Evaluation

This moment is not present in Book B.

<u>Praxeology "Power Function"</u>

Moment #1: First encounter

This type of task is used as an example on how to use the definition of derivatives in a previous chapter. It is later again used as an example but this time to find "a formula for the derivative of $f(x) = x^2$" (Hughes-Hallett, Gleason, et al, 1994, p.115) which is found by approximation (trying different values for x) and using the definition.

The introduction to Chapter 4 called "Short-Cuts to Differentiation" states that derivatives of functions given by formulas will be studied and that the chapter also includes rules to differentiate combination of functions. This gives the motivation of the praxeology, which is to differentiate function using formulas and rules.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

Book B explores this type of task by finding the derivative of $f(x) = x^2$ and $g(x) = x^3$ using the definition of derivative. Furthermore, Book B presents the graph of each function with the graph of its derivative in the same figure to emphasize the relationship between the function and its derivative. Before giving the technique, the authors tell the reader that similar calculations would show that

$$\frac{d}{dx}(x^4) = 4x^3, \text{and} \frac{d}{dx}(x^5) = 5x^4$$

The authors then point out that these results should lead the reader to speculate what the technique is, therefore drawing attention to the emergent pattern.

Moment #4: Work on the technique

At first, Book B states that this technique applies to positive powers of n. Following the presentation of the technique, an example is solved using this technique.

Then, the derivative $\frac{d}{dx}(x^{-2})$ is found using the definition. The graphs of both the function and the derivative function are also provided. After this example, it is stated that this suggests that the technique also works for negative powers. The reader is then told that this is in fact the case and the technique works for all constant real numbers n though, the authors say, they are not in a position to justify it yet. Afterwards, examples using negative and rational exponents are solved using the technique.

An example also asks to find the second derivatives of three functions and to interpret the meaning of those second derivatives. Finally, this technique has to be used in the exercises section where the worked examples can be used as templates to guide the readers.

Praxeology "Derivative of a Constant Times a Function"

Moment #1: First encounter

As previously mentioned, the introduction to Chapter 4 called "Short-Cuts to Differentiation" provides the motivation behind the praxeology.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

The graph of a function $y = f(x)$ is shown as well as the graph of $y = cf(x)$ where $c = 3, \frac{1}{2}$ and -2. Then, Book B explores the relationship between the derivatives of these functions. More specifically, Book B looks at how the graph stretches, shrinks or flips depending on the constant. Most importantly, they explore the change in the slopes of these graphs thus in the derivatives of the functions. The authors conclude with "if a function is multiplied by a constant, c, so is its derivative." (p.187) This then leads to the technique.

Moment #4: Work on the technique

No work on the technique is provided immediately following the presentation of the technique. However, the technique is used in examples later on in combination with other technique to find the derivative of more elaborate functions. As in the case with the previous praxeologies, this technique has to be used in the exercises sections of the chapter.

<u>Praxeology "Derivative of Addition/Subtraction of Functions"</u>

Moment #1: First encounter

In Chapter 2 called "Limits and Derivatives", the type of task of this praxeology is used as a worked example to show how to use the definition of derivative to find the equation of a tangent line to a function, providing the first encounter with the praxeology.

The purpose of the praxeology is found in the introduction to Chapter 4 called "Short-Cuts to Differentiation" which states that rules to find the derivatives of combination of functions will be provided.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

A table lists the values for two arbitrary functions f and g evaluated at integer values of x between 0 and 7. In addition, the values of the sum function formed by $f + g$ are also presented. Then, the sum of the increments of the functions taken separately are compared with the increments of the sum function to come to the conclusion that the rate at which the sum function increases is the same as the rate of the first function plus the rate of the second function. This then leads directly to the technique.

Moment #4: Work on the technique

No examples are provided following the presentation of this technique. However, in a later section of the book called "Derivatives of Polynomials", two examples ask to find the derivative of functions each using the techniques τ_1, τ_2, τ_3 and τ_4.

Another example in the same section asks to find the velocity and acceleration of a body whose position is given by a polynomial function. The velocity is explained to be the derivative of the position and the acceleration to be the derivative of the velocity. Those derivatives are then found using the techniques τ_1, τ_2, τ_3 and τ_4. Lastly, this technique has to be used in the exercises sections of the chapter.

Discussion

The first encounter is present in Book B but not very developed. The introduction to Chapter 4 called "Short-Cuts to Differentiation" provides the motivation behind the praxeologies but only two praxeologies, "power function" and "derivative of addition/subtraction of functions", were previously encountered through a task. For the other four, the exploration of the type of task, when there is one, is the first time the type of task associated with those praxeologies is met.

The second moment is quite interesting in Book B. This book frequently uses graphs to guide the reader to the technique. The praxeology "constant function" uses the fact that the graph of the function is a straight line with slope 0 to develop the technique. Another valuable utilization of graphs of functions is for the praxeology "derivative of a constant times a function". The authors develop the technique from the observation of what happens to the graph of a function when that function is multiplied by different constants. The technique for the differentiation of power functions is developed by exploring the tasks of differentiating $f(x) = x^2$ and $f(x) = x^3$ using the definition. However, the authors still provide the graphs of the functions and of their derivatives and shortly explain the relationship between the two. Another interesting way they develop a technique is for the technique to differentiate a sum of functions. It is simple and easy to understand yet it still uses the interpretation of the derivative as a rate of change. For every praxeology, the techniques are developed and not simply stated.

The third moment is where the theoretical block is built. The theory is built in chapter 2 called "Key Concept: The Derivative". The definition of derivative is presented and an informal definition of limits is also introduced. However, as was mentioned before (see § 2.2.2), the theory is incomplete. There is no mention of the limit laws though they are used in the technologies.

On the other hand, moment #4 – working on the technique – does appear in the text, mostly in the form of worked examples. Most examples are solved directly using the technique. However, two examples involve interpreting the derivative. In one of them, the technique to differentiate a power function is used to find the second derivative of functions. Then, those second derivatives have to be interpreted as to what they mean for the shape of graph of the functions. The second example requires finding the velocity and acceleration of a body. This is accomplished by finding the first and second derivative of a polynomial function using techniques τ_1, τ_2, τ_3 and τ_4. Students are therefore reminded of the different interpretations of the derivative and also learn to find second derivatives.

The fifth moment happens in the introduction to the chapter. Students are told the chapter will develop formulas to differentiate functions and rules to differentiate combination of functions. This points to what the student need to learn in this chapter and what type of tasks they will need to be able to perform at the end of the chapter.

As is the case with book A, the evaluation, moment #6, doesn't take place in Book B either.

3.2.3 Book C

Moment #3: Construction of the theoretical block $[\boldsymbol{\theta}, \boldsymbol{\Theta}]$ ***relative to τ***

The construction of the theoretical block is performed in previous chapters of the book.

Definition:

The definition of derivative is provided and the interpretations of the derivative as the slope of a tangent line to the graph, velocity and rate of change are also explained.

Limits:

Limits are introduced in a preceding chapter. Book C starts with a provisional definition: "The function f approaches the limit l near a, if we can make $f(x)$ as close as we like to l by requiring that x be sufficiently close to, but unequal to, a" (Spivak, 1980, p.84). After many examples using this definition, the $\varepsilon - \delta$ definition is introduced. The limit laws are also stated and proved.

The limit laws as well as the definitions of derivative and limits are integrated into the proofs of the techniques. However, even if the theory is used in this chapter the author doesn't reference it.

Moment #5: Institutionalization

The introduction to chapter 10 "Differentiation" states that this chapter will teach how to differentiate various functions without having to use the definition. It also says that theorems will provide mechanical ways to differentiate functions that are formed from simpler functions by addition, subtraction, multiplication and division, amongst others. Also, the exercises section focus mostly on theoretical knowledge with few computational problems. This implicitly lets the readers know what they need to learn in this chapter and what types of tasks they will need to be able to perform.

Moment #6: Evaluation

This moment is not present in the textbook.

Praxeology "Power Function"

Moment #1: First encounter

The first encounter with the type of task of this praxeology is as an example on how to use the definition of derivative. In addition, this example provided a way to understand the interpretation of the derivative as the slope of a tangent.

The introduction to Chapter 10 called "Differentiation" provides the motivation behind the praxeology. It says that the goal of this chapter is to learn to differentiate a large class of functions without having to use the definition since using the definition can be a laborious process.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

The elaboration of the technique starts with the proof using the definition of derivative that the derivative of $f(x) = x$ is $f'(x) = 1$. A few theorems later, the technique to find $f'(x)$ when $f(x) = x^n$ is stated with a proof by induction on n.

Moment #4: Work on the technique

This moment is not explicit in the book. There is no work on this technique immediately following its presentation. Later on in the chapter examples will be worked out using this technique. However, the work on the technique mostly takes place in the exercises section of the textbook and the readers have to complete this moment on their own.

Praxeology "Derivative of Addition/Subtraction of Functions"

Moment #1: First encounter

The first encounter with this praxeology is as a worked example on how to use the definition of derivative. This example, $f(x) = cx + b$, also serves to demonstrate the interpretation of the derivative as the slope of a tangent line to the graph. Furthermore, as previously mentioned, the introduction to the chapter provides the motive behind the praxeology.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

This moment doesn't take place in Book C.

Moment #4: Work on the technique

The technique to find the derivative of the sum of two functions is presented before the technique to find the derivative of a function multiplied by a constant. Therefore, following the presentation and proof of this latter technique, the author makes note that $(-f)'(a) = -f'(a)$ therefore $(f - g)'(a) =$

$(f + [-g])'(a) = f'(a) - g'(a)$, consequently extending the technique for sum of functions to difference of functions.

Book C provides no numerical example to work on this technique. However, it shows how this technique combined with other techniques could be used to differentiate any function of the form $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$

where $f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + 2 a_2 x + a_1$

and $f''(x) = n(n-1) a_n x^{n-2} + (n-1)(n-2) a_{n-1} x^{n-3} + \cdots + 2 a_2$

There are also a few problems using this technique in the exercise section of the textbook. In fact, this is where most of the work on the technique will take place, mainly through problems about the chain rule or about rates of change.

<u>Praxeology "Derivative of a Quotient of Functions"</u>

Moment #1: First encounter

The introduction to Chapter 10 called "Differentiation" provides the motivation behind the praxeology, as stated previously.

Moment #2: Exploration of the type of task T and Elaboration of a technique τ relative to T

This technique is split into two theorems. The first theorem provides part of the elaboration of the technique, showing that $\left(\frac{1}{g}\right)'(a) = \frac{-g'(a)}{[g(a)]^2}$ using the definition of derivative. The technique for the quotient of functions is then proved using the technique to find the derivative of a product of functions. Since $\frac{f}{g} = f(\frac{1}{g})$, $\left(\frac{f}{g}\right)'(a) = \left(f\ \frac{1}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$. No further examples or exercises are provided to explore this type of task.

Moment #4: Work on the technique

Some worked examples are provided where the technique is used. Other techniques are also used in those examples, more specifically the techniques for the praxeology “constant function”, “power function” and “derivative of addition/subtraction of functions”.

Most of the work on the technique has to be done in the exercises. Therefore, the readers have to deal with this moment mostly by themselves as they perform the exercises. Also, most of these exercises won’t be about the techniques themselves but the techniques will have to be used to find a rate of change or to differentiate a function using the chain rule, for example. Hence, this moment is indeed present in the book though implicitly.

Discussion

The introduction to Chapter 10 “Differentiation” provides the motivation behind the praxeology which is part of the first encounter. Also, the previous chapter called “Derivatives” provided a few worked examples which were the first encounter with the type of tasks of the praxeologies “constant function”, “power function” and “derivative of addition/subtraction of functions”.

The second moment is missing for most of the praxeologies presented here. The only two praxeologies where there is an elaboration of the technique is for the differentiation of a power function and for the differentiation of a quotient of function. And this is only because those techniques were split into two parts. Before presenting the technique to differentiate a power function, Book C presents the technique to differentiate $f(x) = x$. Similarly, the technique for the differentiation of the quotient of function is offered after the technique to differentiate $f(x) = \frac{1}{g(x)}$ has been presented.

The third moment is the construction of the theoretical block. Book C presents a complete theory in the chapters preceding the chapter where the praxeologies are introduced. The definition of derivative and of limit as well as the limit laws are all presented, explained and proved.

Moment #4 is not explicit in this textbook. All the techniques are used in the exercises section at the end of the chapter however there are very few worked examples. Those worked examples are to work on the techniques for the product and quotient of functions. The examples for the "derivative of a product of functions" praxeology involve only trigonometric functions which is a way to introduce the readers to the formulas of the derivatives for this kind of functions. Book C also provides a non numerical example to show the reader how to differentiate polynomials. All those examples are solved using the given techniques. An interesting part of the book is when the technique to differentiate a function multiplied by a constant and the technique to differentiate a sum of functions are used to show how to differentiate a difference of functions. Most of moment #4, however, takes place in the exercises section of the textbook therefore this moment has to be performed by the readers. Moreover, even though the techniques are used in the exercises, usually the problems are not about those techniques themselves. For instance, the techniques may be used in an exercise about the chain rule or in a problem about rate of change. However, there are two exercises where the techniques of the praxeologies shown here are the main techniques used. The first one involve finding $f'\big(f(x)\big)$ for given functions f and the second one asks to find $f(f'(x))$ for different functions f.

Institutionalization happens in the introduction of Chapter 10 called "Differentiation". The introduction tells the students that by the end of that chapter, they should be able to differentiate functions formed by the addition, subtraction, multiplication and division of other functions. Also, they know that they have to learn the rules to differentiate a large number of functions without having to use the definition of derivative. This tells them what the praxeologies presented are about; though they are reminded that using the definition of derivative may sometime be the only possible approach.

Finally, as is the case with Book A and B, moment #6 is not present in this textbook.

3.3 Discussion

The three textbooks provide the motivation behind the praxeology in the introduction. However, the first encounter is more than that and should usually happen through a type of tasks. In every book, some tasks were previously encountered in the sections where the books introduced the definition of derivatives. Furthermore, every book had presented worked examples either using the power or constant functions or functions formed by other functions through addition, subtraction, multiplication or division as a way to help the students see the derivative as the slope of a tangent line to a graph or as a rate of change in previous sections of the books. Book A is the book where there are the most praxeologies where the first encounter happens through a type of tasks. Only the praxeologies "constant function" and "derivative of a product of functions" don't provide this kind of first encounter. Book B, on the other hand, provides a first encounter through a type of task for only two praxeologies and Book C for three.

The second moment is where the types of tasks are explored and the techniques are elaborated. Book C is clearly lacking this moment. Only twice is there an elaboration of a technique and those happen only because the techniques have been split into two parts. Usually, book C merely states the techniques then proves them. On the other hand, Book B carefully develops every technique. Usually this elaboration involves graphs but also some algebraic manipulations as in the case for the technique for the differentiation of a product of functions. In between these two books, there is Book A. Two techniques are presented without elaboration or exploration. However, the others are developed either graphically and geometrically for the constant function and for the product and quotient of functions or using the definition for the power function. Hence, Book A and B take the time to explore the types of tasks and develop the techniques using the different interpretations of derivatives while Book C simply state the techniques with their proofs.

Moment #3 is a very important moment in the learning process as this is when the knowledge block is built. However, unlike Book A and C, Book B doesn't put together all the theory necessary to support the techniques and technologies. Limit laws are not presented nor is the $\varepsilon - \delta$definition of limit; only an informal definition is offered.

The work on the technique is when the technique is tested through problems so that it can be retouched if necessary (moment #4). Book C is different from the other two textbooks as this moment is not explicitly present. This moment mostly takes place in the exercises section of the textbook so the readers have to perform that moment by themselves. Also, these exercises usually focus on the chain rule and the interpretation of the derivative as a rate of change so the techniques are of secondary importance. On the other hand, the other two books provide a lot of worked examples where techniques are applied therefore the moment is explicit. Most of those problems are simply computational problems but some of them are word problems involving the interpretation of the derivative as the slope of a tangent line to a graph and as a rate of change. Book B goes further and work with the second derivative and what it says about the graph of a function and how it can be interpreted as acceleration. The exercises section of Books A and B could also be used to work on the technique. However, the exercises in Book A mostly involve using the worked examples as templates making these types of problems into routine tasks. The worked examples in Book B can also be used as templates to solve exercises however, the exercises often also require the readers to explain their results in words or using a graph and this is where the worked examples cannot be used as templates anymore.

The fifth moment is the moment of the institutionalization. This is the moment that tells the students what is exactly the praxeology that they are expected to learn. This moment starts in the introduction of the chapter for all three textbooks. They basically tell the students that they should learn the processes for differentiating functions formed by other functions through addition, subtraction, multiplication and division – thus institutionalizing only the know-how block of the praxeologies. The introduction also tells them to learn how to differentiate various functions for which there are special rules. All of this should be accomplished without using the definition. The worked examples and exercises also tell the readers what knowledge is institutionally relevant. Books A and B focus on computational knowledge. Moreover, Book A also focuses on the interpretations of the derivative as a rate of change and as the slope of the tangent line to a graph while the examples and exercises in Book B focus on the graphical interpretations. Finally, the examples and exercises in Book C focus on theoretical knowledge.

Finally, moment #6 doesn't appear in any of the books.

The table below shows the occurrences of the didactic moments in the three analyzed textbooks. In the particular case of moment #4, a distinction between an "explicit" and an "implicit" occurrence is made, referring to the occurrence of worked examples vs. a work on the technique mostly left to the reader (in the exercises section), respectively.

Book/Moment	#1	#2	#3	#4	#5	#6
Book A	Present	Absent for 2 praxeologies	Present	Explicit	Present	Absent
Book B	Present	Present	Present – no limit laws	Explicit	Present	Absent
Book C	Present	Absent for 4 praxeologies	Present	Implicit	Present	Absent

Table 2. Presence/absence of the didactic moments in the three textbooks.

4. Model Readers

4.1 Theoretical Framework

In addition to looking into the textbooks' organization of mathematical knowledge and the presence in these textbooks of the didactic moments from the perspective of the Anthropological Theory of the Didactic, I explored the ways in which the textbooks could format the students' relationship with mathematical knowledge. To accomplish this I used tools developed by Sierpinska (1997) (see Bruner, 1985; Voigt, 1995).

The text will assume certain *competences* of its reader and will also devote itself to the creation of other competences in this reader. By assuming certain competence on the part of the reader, the text puts constraints on how it will be interpreted. By doing so, it channels the reader into an understanding of the text that is within its intended interpretation.

> "The Model Reader, therefore, erects boundaries to the universe of the *interpretation* of the text. These boundaries also imply certain limits on how the text can be used. Going beyond these limits forces a change in the universe of interpretation." (Sierpinska, 1997, p.5)

To further analyze the model reader, I will use Weinberg's description of two different readers of a textbook (Weinberg and Wiesner, 2010). There is the *intended reader* and the *implied reader*. These two readers will then form the model reader. "The *intended reader* is the image of the reader that the author form in his or her mind, the *implied reader* is the collection of qualities required of the empirical reader in order to correctly interpret the text..." (Weinberg and Wiesner, 2010, p.51)

Weinberg and Wiesner's framework define also the empirical reader, an individual who actually reads the book. In this research, the empirical reader cannot be described here with certainty. For the analysis, I will make the assumption that the empirical reader is a student of college calculus.

4.2 Model Reader in the Three Textbooks

This section looks at how the three chosen textbooks may format the student-reader relationship with calculus knowledge. For instance, the text can format the way the students will interpret the concepts as well as which competencies will be developed in the students. In what follows, I analyze the Model Reader of each of the three textbooks.

4.2.1 Book A

The first place to look at to get a better understanding of the Model Reader is the preface. This is where the author's purpose is often stated which can establish constraints on the interpretation of the text as well as make clear what competence the author wants to develop in the reader. Therefore, the preface will shed light on the intended reader. In the case of Book A, the author's purpose is made clear in the preface where he expresses that he wants to: "convey to the student a sense of the utility of calculus and develop technical competence, but I also strive to give some appreciation for the intrinsic beauty of the subject." (Stewart, 2003, p. xiii) While the second part of his assertion is difficult to judge, the first part gives an idea as to what the reader is expected to accomplish. As he states later on in the preface, his focus is to make students understand concepts. He claims he will accomplish this using the Rule of Three, which is to present the topics geometrically, numerically and algebraically. Throughout the book, the author keeps his promise to present every topic in those three ways. In the chapter that was of most interest to me about the algebra of differentiation and rules for special functions, those three ways were used with a focus on presenting the topics algebraically. According to Weinberg and Wiesner (2010), the number of worked examples indicates that the intended reader is someone who can interpret the textbook as presented but may need assistance as the presentation of the topics alone would not be sufficient. Indeed, Stewart has a lot of worked examples which would in general indicate that "the authors anticipate a reader who requires more guidance and prompting than the reader of an academic text." (Weinberg and Wiesner, 2010, p.51) Furthermore, we get another glimpse at the intended reader from the address to the students the author wrote. An important point made is that the intended reader reads the text and understands it before doing the exercises. (Stewart, 2003, p. xxiv) The intended reader therefore is someone who is capable of performing such actions of understanding the material and interpreting it to perform the required exercises. Furthermore, the intended reader will frequently work with technology; the textbook devotes a section to this topic, called "Graphing Calculators and Computers". The intended reader therefore does not already have the ability to work with technology as the author makes sure the necessary skills are acquired.

As mentioned previously, the implied reader possesses all the competencies that the reader is expected to possess to be able to correctly interpret the text. One of those competence required in Book A is algebra. For instance, all calculations presented in the text are devoid of algebraic explanations. In the preface, the author said that he would strive to present the mathematical topics in three ways, one of them being algebraic. This leads to an implied reader who has an algebraic background strong enough to understand the proofs and the examples without guidance from the text. The implied reader also needs to have understood topics previously presented in the book to be able acquire new concepts. Hence, he needs to be able to make links to previously learned mathematical notions. This characteristic can be seen frequently in the textbook. The author rarely reference the theory in the justification of the techniques thus assuming the reader has already assimilated the theory and is able to make the connection by himself or herself. For example, in the praxeology "constant function", there are two technologies presented, one of which simply states that since the graph of the function is a horizontal line with slope zero, the derivative is 0. The reader has to make the connection with the previously presented interpretation of the derivative as the slope of a tangent line to a graph and deduce that since the graph has slope 0 everywhere, the tangent line to this graph would also have a slope of 0 everywhere. The only theory referenced in the praxeologies examined here are the limit laws. Also, the implied reader is expected to understand the mathematical language and concepts used in the textbook that are not defined. For instance, the tangent line is very present in the chapter 3 called "Differentiation Rules" where the praxeologies studied are presented. However, the definition of a tangent line is never provided in the textbook.

Discussion

The preface of the text states what competences the intended reader will have developed after reading the textbook. The author wants the reader to develop a technical competence and to understand concepts. The intended reader will also be able to understand graphical, numerical and algebraic interpretations of the material as this is how the author aims at presenting every topic. Finally, the intended reader should be able to interpret the text to attain the level of understanding necessary to do the exercises.

The implied reader refers to what competences and knowledge should the reader already possess to properly respond to the text. First, as discussed above, the reader should have some background in algebra as the algebraic manipulations in the text are never explained. Also, since the theory is rarely referenced, the readers have to make those connections themselves to interpret the justifications in a meaningful way. Some very important words for the praxeologies, tangent and rate of change, are not defined in Book A. However, to get a good understanding of the text, those words have to be understood therefore this knowledge have to be acquired outside of Book A.

4.2.2 Book B

To discover what is the model reader in this book, I again started by looking at the preface. The preface clearly tells what the authors want to achieve and the competences they want to develop in the intended reader. The authors value the Rule of Three and strive to present every topic numerically, geometrically and algebraically. They state that they will always encourage the reader to think about the geometrical and numerical interpretation of the topics they are presenting to reinforce the algebraic aspect of calculus. Indeed, the authors don't want to "undermine the purely algebraic aspect of calculus, but rather reinforce it by giving meaning to the symbol." (Hughes, Hallett, Gleason, et al., 1994, p. vii) As could be expected, looking throughout the text and especially in the chapters about derivatives, I found that the algebra involved in this book is not central and the focus is more on the graphical aspect. To reinforce that algebra is not prevalent in the book, the authors mention that "the curriculum is thought-provoking for well prepared students while still accessible to students with weak algebra backgrounds." Also, in building the curriculum the authors claim they consulted engineers, chemists, mathematicians, etc and adapted it according to those specialists' comments. The focus is therefore on real life and numerical situations. Moreover, the intended reader of this book will be able to accomplish the actions that the authors, in the address to the students, say are necessary to succeed which is "reading, questioning, and thinking hard about the ideas presented. It will be helpful to read the text in detail, not just the worked examples." (Hughes, Hallett, Gleason, et al., 1994, p. xiii) Furthermore, as previously mentioned, Weinberg and Wiesner's interpretation of the many worked examples is that the authors expect the reader to need guidance throughout the text. (Weinberg and Wiesner, 2010, p.51) To be able to solve the exercises, the authors feel the students need examples.

The implied reader doesn't need a deep background in algebra as can be understood from the quote in the previous paragraph. Most worked examples are very detailed making sure the algebra and arithmetic performed are well understood as can be seen from the example below:

Find $\frac{d}{dx}(x^{-2})$

Solution: Let $k(x) = x^{-2}$. Then, provided $x \neq 0$,

$$\frac{d}{dx}(x^{-2}) = \frac{d}{dx}\left(\frac{1}{x^2}\right) = \lim_{h\to 0}\frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x+h)^2}{(x+h)^2x^2}\right] \text{ Combining fractions over common denominator}$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{x^2-(x^2+2xh+h^2)}{(x+h)^2x^2}\right] \text{ Multiplying out}$$

$$= \lim_{h\to 0}\frac{-2xh-h^2}{h(x+h)^2x^2} \text{ Simplifying numerator}$$

$$= \lim_{h\to 0}\frac{-2x-h}{(x+h)^2x^2} \text{ Dividing numerator and denominator by } h$$

$$= \frac{-2x}{x^4} \text{ Letting } h \to 0$$

$$= -\frac{2}{x^3} = -2x^{-3}$$

(Hughes, Hallett, Gleason, et al., 1994, p. 193)

This textbook, as pointed out in the previous paragraph, focuses on graphical but also on numerical interpretation. Therefore, the many real life examples require the reader to be able to understand the contexts of those examples, be it from the business world or the science world. For example, in the exercises section of the chapter where the praxeologies studied were presented, the following problem is offered to the readers (Hughes, Hallett, Gleason, et al., 1994, p. 198):

42. The period, T, of a pendulum is given in terms of its length, l, by

$$T = 2\pi\sqrt{\frac{l}{g}}$$

where g is the acceleration due to gravity (a constant).

(a) Find $\frac{dT}{dl}$.

(b) What is the sign of $\frac{dT}{dl}$? What does this tell you about the period of the pendulums?

The intended reader knows what a pendulum is but most importantly what its period represent as well as how the gravity and its length impact on its period. Also, the implied reader understands graphical representations of concepts as it is the most widely employed tool to represent them. For instance, in the previous chapter it was pointed out that Book B frequently uses the graphical interpretation of the derivative to elaborate the techniques (see § 3.2.2). Furthermore, important mathematical terms in the context of differentiation have to be known by the implied reader. One such term is tangent line and is not defined in Book B. Finally, the reader has to be able to make connections to previously learnt concepts. The theory is never referenced in the justifications of the techniques therefore the reader has to have assimilated the theory to correctly understand the techniques.

Discussion

The authors want the reader to understand what the concepts mean graphically and numerically to reinforce the algebraic interpretation. Therefore, the intended reader will develop an understanding of the concepts beyond the algebraic interpretation. The intended reader will be capable of thinking about and questioning the ideas presented. On the other hand, he or she may need guidance beyond the explanation of the concepts as is suggested by the number of worked examples.

The implied reader doesn't have a strong background in algebra as can be understood from the care the authors made on explaining most algebraic manipulations done in the book. However, they are expected to understand different concepts from real life situation to be able to follow the examples and to be able to solve the problems in the exercises section. The extended use of graphs in the different didactic moments, most importantly moment #2 and moment #4, indicates that the reader should be comfortable working with various graphs. Finally, the lack of reference to the theory shows that the reader has to be able to make the connections by himself or herself to interpret the text meaningfully.

4.2.3 Book C

In the preface, the author states that the problems in the book have been designed for "students who have seen a small amount of calculus previously, and for bright students with a reasonable background." (Spivak, 1980, p.vii) This clearly defines who the intended reader is. Also, the problems and examples are never overly simple and sometimes quite complex. For instance, the examples to work on the technique for the praxeology "derivative of a product of functions" use only trigonometric functions. Those have never been encountered before so Book C provides the rules to differentiate such functions followed by examples. Book C aims at persuading the students that rigor and precision are "the natural medium in which to formulate and think about mathematical questions." (Spivak, 1980, p.vii) The author's goal is not only to have the student understand the concepts presented but also to appreciate the rigor with which they are presented. Working with that much rigor is not within the reach of every student thus the need for the students to already have a background in calculus or to have a certain facility when it comes to mathematics. Still in the preface, the author says: "In addition to developing the students' intuition about the beautiful concepts of analysis, it is surely equally important to persuade them that precision and rigor are neither deterrents to intuition, not ends in themselves, but the natural medium in which to formulate and think about mathematical questions." (Spivak, 1980, p. vii) The intended reader will therefore develop a rigorous approach to calculus and mathematics in general. Also, the book contains very few worked examples and those are never simple implying that the author expects the readers to be able to navigate the concepts by themselves and not to need worked examples to support them.

The implied reader in this book is able to solve the exercises simply by applying the concepts taught and not by mimicking what was done in a worked example. Also, looking at the didactic moments, it is clear that the reader has to be autonomous in many of those moments. Moment #2, when the reader should explore the type of task and elaborate a technique is almost completely missing in Book C. Moreover, the work on the technique is left to the exercises where the student has to do that work without guidance from the textbook. Those exercises require the use of the theory which has rarely been mentioned in the chapter. Indeed, there is no reference to the theory in the justification of techniques so the reader has to make those connections alone to understand the proofs in a meaningful way. This means an implied reader who is not only able to make connection between the technology and the theory but also to work on the technique using the theory without support. Also, there are no explanations accompanying the examples or proofs consequently, algebraic calculations should be easy to perform and following a rigorous proof is within the ability of the implied reader.

Discussion

Book C's intended reader is very clear from the preface. The author states what the reader should accomplish and who that reader should be. He or she has a strong mathematical background and is able to understand the concepts presented with rigor and precision. The intended reader will not only understand the concepts but will also develop a rigorous approach to mathematics in general. As before, the fact that few worked examples are provided is interpreted as defining an intended reader who doesn't need guidance to understand the concepts.

The implied reader is able to make the connections between the technologies and the theory. The reader can also work out the examples without having a framework such as worked examples to work with. Furthermore, the implied reader will be able to understand the techniques and where they come from by themselves, without needing an exploration of the tasks or an elaboration of the techniques. Lastly, the implied reader is able to perform the work on the techniques solely from the exercises that are very theory oriented with such theory having been rarely referenced in the chapter.

4.3 Discussion

The intended reader is very different in the three books. Book C is meant for an experimented reader while Book B has been designed so that readers with weak background in algebra could follow. Book A, on the other hand, stands somewhat in between Books B and C though the author doesn't mention it as clearly as the other two. From the address to the students, the intended reader of Book A is understood to able to understand the text and interpret it as presented. Therefore, the lack of algebraic explanations in the text indicates that the intended reader has a strong enough background to perform these tasks without assistance. However, Book A is nowhere near as rigorous as book C. All three books aim at making the student understand the concepts of calculus however they all do so in a different manner. Book C emphasizes the importance of rigor and precision and the intended reader of this textbook will come to think about mathematic in a rigorous way. Book B focuses on graphical and numerical interpretations of the concepts to support the algebraic interpretation. As a result, the intended reader will develop different competencies than the intended reader of Book C. Books A and B aim at developing technical understanding geometrically, numerically and algebraically. However, Book B is more geometrical and numerical and less algebraic than Book A. As a final remark about the intended reader, I point out that in Books A and B, the intended reader is one who needs guidance in understanding the concepts (as the high number of worked examples suggests). The intended reader in Book C does not need this type of guidance.

The implied reader is also quite different from book to book. Both Book A and Book C require an algebraic background as the algebraic manipulations are never explained. On the other hand, Book B carefully explains every step of examples and proofs when algebra is used. All three books require the reader to be able to make connections with previously learnt material as the theory is never referenced for both Book B and C and only when limit laws are used for Book A. Furthermore, Book C's implied reader is able to do the work on the technique almost entirely in the exercise section without worked examples as a guide. The exploration of the types of task and the elaboration of the techniques is a moment that is mostly absent in Book C so it again falls on the reader to fill in the void therefore, the reader needs to have the capacity to do so. A very important concept for the understanding of the derivative is the tangent line to a graph however it is never defined in Book A and B. In contrast, Book C provides a complete definition involving secant lines and taking the limit of those lines. This suggests that unlike Book C, Books A and B expect the reader to have already acquired that knowledge.

5. Didactic Layer

The didactic layer is everything that is meant to shape the readers' interpretation of the text as well as shape how the reader uses the text. In this chapter, I will take a closer look at the three textbooks' didactic layers by looking at the textbooks' address to the students section (when one is presented), the examples and exercises and any other indicators of the authors' attempt to shape the usage and interpretation of the text.

5.1 Book A

Following the preface, the author addresses the readers directly formatting how they will use the textbook. For instance, he tells them to read and understand the concepts before doing the exercises and to do those exercises in a complete and logical fashion instead of writing disconnected equations and formulas. (Stewart, 2003, p.xxiv) He also explains the various symbols he uses throughout the book. For example, he has a specific symbol to warn against making some "common" mistakes as well as a symbol to identify exercises that will require the use of a graphing device. The "common mistakes" symbol is frequently used and shapes how the reader is expected to interpret the text or more exactly how the text should not be interpreted. For instance, before deriving the technique to differentiate the product of two functions, the author states that even though the reader might be tempted to guess that the derivative of the product of two functions is the product of their derivatives, it is not the case. He even illustrates this common mistake by calculating the derivative of $f(x) = xx^2 = x^3$ using the power rule to demonstrate that it is not equal to the derivative of x multiplied by the derivative of x^2. The author also attempts to correct misinterpretations in the way he constructed the exercises. Indeed, he will sometimes ask the readers to execute the same exercises using two different techniques to show that they are equivalent or he might require that the readers check the validity of their answers using a graphing device. This shows that he is aware that there might be different interpretations and also that different behaviors or competences might have led the readers on different paths to get to the same answer. Furthermore, the author also formats how the mathematical layer should be interpreted by making sure the steps of the justifications of the technologies are well understood as well as the solutions of the examples. For instance, when a limit law is used in a justification of a technique, the law is usually mentioned in parenthesis with a number. This number refers to the number of the law as numbered in a previous section. However, sometimes limit laws are used and not referenced which might undermine the authors attempt to format the readers' behavior in front of those tasks. A more specific example of formatting would be a worked example where the derivative of $y = \sqrt[3]{x^2}$ is found (Stewart, 2003, Chapter 3, section 1, p.186). A side note shows the graph of both the function and its derivative. It also asks the reader to notice that y is not differentiable at 0 as y' is not defined at that point. This note could be problematic for readers trying to interpret it from the point of view of the derivative being the slope of the tangent line since this function does have a tangent line at $x = 0$ even though the function is not differentiable at that point. Furthermore, this does not abide by the definition of differentiability of a function, given in a previous chapter of the book (Chapter 2, section 9, p. 170). This is interesting as there are also a few problems in the exercises section of Chapter 3, section 1, where the reader is asked to determine if a function is differentiable or where it is differentiable. Considering the worked out examples and the mathematics discourse in this section, we can assume that the reader is expected to deal with these exercises either by comparing the domain of the function with the domain of the derivative function or by informally looking at the graph of the function. In the first case, the reader is not expected to use the definition of differentiability (previously given); but the link between these two approaches to differentiability is never addressed in the textbook (except for the above mentioned example). It seems that the author wants this technique to decide differentiability – comparing the domain of the function with the domain of the derivative function – to be part of what the readers will retain. The technique, however, is only presented in an example, and there is not theoretical discourse (not even the statement of a theorem) justifying the validity of the technique. In the second case (analyzing the graph of the function to decide differentiability), the technique is informally stated and informally justified: "In general, if the graph of a function f has a "corner" or "kink" in it, then the graph of f has no tangent at this point and f is not differentiable there. [In trying to compute $f'(a)$, we find that the left and right limits are different.]" (Stewart, 2003, p. 172)

The examples and exercises are also used to shape how the mathematical layer is to be interpreted and applied. For instance, one example is solved using two different techniques in the praxeology "product of functions" (Stewart, 2003, p. 194):

Differentiate the function $f(t) = \sqrt{t}(1-t)$.

Solution 1 Using the Product Rule, we have

$$f'(t) = \sqrt{t}\frac{d}{dt}(1-t) + (1-t)\frac{d}{dt}\sqrt{t}$$

$$= \sqrt{t}(-1) + (1-t)\left(\frac{1}{2}t^{-\frac{1}{2}}\right)$$

$$= -\sqrt{t} + \frac{1-t}{2\sqrt{t}} = \frac{1-3t}{2\sqrt{t}}$$

Solution 2 If we first use the laws of exponents to rewrite $f(t)$, then we can proceed directly without using the Product Rule.

$$f(t) = \sqrt{t} - t\sqrt{t} = t^{\frac{1}{2}} - t^{\frac{3}{2}}$$

$$f'(t) = \frac{1}{2}t^{-\frac{1}{2}} - \frac{3}{2}t^{1/2}$$

which is equivalent to the answer given in Solution 1.

Following this example, the author states that this example “shows that it is sometimes easier to simplify a product of functions than to use the product rule.” (Stewart, 2003, p. 194) Afterwards, he does state that sometimes the product rule is the only approach but this first comment certainly is an attempt to shape how the readers will approach the differentiation of a function when a multiplication is involved. To further emphasis this recommendation, two problems in the exercises section of the textbook ask the readers to differentiate a function in two different ways: first by simplifying the function and then by using the product or quotient rule directly. Book A’s examples are varied in the level of difficulty and in the type of questions asked. After each technique, except for the constant function, there are examples where the techniques are applied to find the derivative of a variety of functions. This serves as a guide to the readers on how to correctly use the techniques. None of those examples reference the definition in any way though in one instance, the derivative of a power function is found using the definition. However, this example occurs as an exploration of the type of task and as an elaboration of the technique (didactic moment #2) and not as a worked example on how to find the derivative of a power function. In addition to the computational examples, there is another type of examples present in Book A. Those are examples where the readers have to remember the interpretations of the derivative as the slope of a tangent line to the graph of a function or as a rate of change. For instance, following the presentation of the technique to differentiate the power function, the readers are asked to find the equation of a line tangent to the graph of a given power function. In those words problems as well as in the numerical problems, the solutions involve using the techniques previously introduced. Those examples shape how the readers will view the mathematical layer as they focus on computation and the different interpretations of the derivative. This is reinforced by the types of problems presented in the exercises section of the textbook. It shows what the author wants the readers to retain from the text, therefore implicitly belonging to the didactic moment of institutionalization. Roughly half of the exercises presented involve differentiating a function to find its derivative using one or more of the techniques previously introduced. The majority of the other problems focus on the interpretation of the derivative as the slope of the tangent line to a graph. Only one problem involves the definition of the derivative (Stewart, 2003, p. 192):

> Use the definition of a derivative to show that if $f(x) = 1/x$, then $f'(x) = -1/x^2$. (This proves the Power Rule for the case $n = -1$).

This problem, however, is a follow up on a worked example from the praxeology “power function”. The function $f(x) = 1/x$ was differentiated using the technique for power functions. At this point, however, this technique had only been proven for positive integer exponents. Following this example, the author said that the proof that the derivative of f is $f'(x) = -1x^{-2}$ is left for the exercises section of the book which brings us back to the only problem in the exercise section involving the definition.

Discussion

Book A dedicates a few pages at the beginning of the book to try formatting how the readers will use the textbook. The author tells the readers that they should focus on understanding the concepts before doing the exercises and he even states how those exercises should be solved. The author also tries to format how the mathematical layer is to be interpreted. He has a symbol to alert the readers of common errors thus attempting to correct misinterpretations of the text (the fact that this symbol may *introduce* mistakes that were not there in the first place is discussed in the conclusions section (Chapter 6)). Book A also makes sure that the text is well understood by explaining some of the steps of the justifications of techniques. More specifically, the author regularly references the limit laws thus making sure those steps are not misinterpreted. Furthermore, he also lets the readers know about a way to avoid complications with the concepts when he tells them that simplifying a function by multiplying it out instead of using the technique for the differentiation of a product of functions is sometimes simpler and more efficient. Both the examples and the exercises focus on applying the techniques to differentiate various functions and on interpreting the derivative as a rate of change or as the slope of a tangent line thus implicitly telling the readers what part of the mathematical layer they should concentrate on.

5.2 Book B

This book has an important didactic layer supporting the mathematical layer. In the preface, there is a section dedicated to the readers on how to use the book thus formatting the usage of the text. For instance, the readers are told that there are few examples in the book that will be similar to the homework problems "so homework problems can't be done by searching for similar-looking "worked out" examples. Success with the homework will come by grappling with the ideas of calculus." (Hughes-Hallett, Gleason, et al., 1994, p.xiii). The readers are also told that they will often be asked to explain their ideas in words or using graphs. Those open-ended questions usually require the readers to use the different interpretations of the derivatives. The authors made sure that the readers perfectly understand all the steps of the solutions and proofs. This will probably correct various misinterpretations. For instance, every steps of a worked example to find the derivative of x^3 is carefully explained (Hughes-Hallett, Gleason, et al., 1994, p.191):

Find the derivative of $g(x) = x^3$.

Solution

$$g'(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{(x+h)^3-x^3}{h}$$

$$\textit{Multiplying out} \rightarrow = \lim_{h\to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h}$$

$$\textit{Dividing by } h \rightarrow = \lim_{h\to 0}(3x^2 + 3xh + h^2) = 3x^2 \leftarrow \textit{Looking at what happens as } h \rightarrow 0$$

The readers are also asked to make links with other concepts previously presented in the books. For example, in the section where the readers learn how to differentiate a function times a constant (Chapter 4, section 1), the exploration of the type of task and the elaboration of the technique involves looking at what happens when a function is multiplied by a constant, more specifically what happens to its graph. This involves getting the readers to refer to what has been previously learnt about the graph of a function but also focus the attention on the graphical interpretation of the derivative. This interpretation will often come back especially in the didactic moment #2. The authors, by emphasising this interpretation, attempt to shape how the readers will interpret the derivative.

Most of the examples provided are basic numerical examples where the technique is applied to find the derivative of a function. However, some of those examples, like the example shown in the previous paragraph, are illustrated with the graph of the function and the graph of the derivative function. There are also examples that focus solely on the graphical interpretation of the derivative. For instance, one example asks to find and interpret the second derivative of three power functions. This example is worked out by determining where the graphs would be concave up and where they would be concave down. This focus on the graphical interpretation strongly influences which part of the mathematical layer the readers will have to give more importance to. This textbook also provides examples that are about other interpretations of the derivatives such as velocity and acceleration. Some examples, however, use the definition of derivative; those examples are part of the exploration of the type of task. One exception being the example where $\frac{d}{dx}(x^{-2})$ is found using the definition. The justification of the technique to differentiating power functions was given only for positive integer powers so this example is used as an opportunity for the authors to say that the technique also holds for negative powers (without proving it). Then, in the exercises section of the textbook, one problem asks the readers to use the definition of derivative to justify the technique for negative integer powers. Those are the only two instances where the definition of derivative is mentioned therefore shaping the level of importance the readers will give to the definition. As mentioned above, the book frequently asks the readers to explain ideas in the exercises thus prompting for an understanding that goes beyond a computational understanding. Most often, those explanations involve explaining what the results imply graphically and occasionally the exercises also require interpreting the derivative as a rate of change. What those problems don't require is using the theoretical block. The exercises section also focuses mostly on the application of the techniques. This is similar to what is found in the worked examples as those are either computational or, as mentioned above, require shortly explaining the graphical implications of the derivative. The readers will therefore know what type of problems to concentrate on. Finally, similarly to Book A, Book B alerts the readers of the possibility of simplifying a function instead of using the technique to differentiate a product of functions. This happens in the exercises section of the textbook. For example, one problem asks: "If $f(x) = x^2(x^3 + 5)$, find $f'(x)$ two ways: by using the product rule and by multiplying out. Do you get the same result? Should you?" (Hughes-Hallett, Gleason, et al., 1994, p.210) Moreover, for some computational problems, the book simply asks to find the derivative though it mentions that in some cases, it might be advantageous to simplify the functions first. These exercises shape how the readers will interpret the mathematical layer.

Discussion

Following the preface, the author addresses the readers directly to tell them how to use the textbook to be successful. The readers are told that they have to focus on understanding the concepts as few exercises will be similar to the worked examples and that the exercises will often require them to explain ideas in words or using a graph. These comments to the readers shape how they will use the textbook, and in particular how they will interpret the mathematical layer. Indeed, there is an emphasis on graphical interpretation in what corresponds to the didactic moment #2 and in the examples and exercises therefore implicitly telling the students what they should concentrate on. Furthermore, the exercises section is mostly made up of problems to practice using the techniques and on problems involving different interpretations of the derivative again implicitly letting the readers know what concepts are most *important*. Also, the authors attempt to correct possible misinterpretations of the text by making sure every step of the examples or technologies are carefully explained. Those explanations focus on the algebraic manipulations and on referencing differentiation techniques previously introduced.

5.3 Book C

The didactic layer is relatively thin in Book C. There is no section where the author tells the readers how to use the text in order to succeed. Nevertheless, the author still attempts to shape how the readers will employ the text. An example of these attempts is when the author introduces the technique to differentiate a quotient of functions. The author states about the technique: “Though not particularly appealing, it is important, and must simply be memorized (I always use the incantation: “bottom times derivative of top, minus top times derivative of bottom, over bottom squared.”)” (Spivak, 1980, p.157) The author not only shapes how the readers should treat the technique but he also gives a way to memorize it. The author also works on shaping the interpretation of the mathematical layer. For instance, he makes sure that the justifications of the techniques are well understood and makes reference to previously learnt concepts. For instance, in the justification of the technique to differentiate $f(x) = 1/g(x)$, the author mentions that it is necessary to check that $\left(\frac{1}{g}\right)(a+h)$ is defined for small h. This requires using the concept of continuity explained in a previous section and the theorem: “If f is differentiable at a, then f is continuous at a.” (Spivak, 1980, p.144) Following the proof of the technique, the author tells the readers to note that once again, he uses the continuity of g at a making sure to correct any misinterpretations that might have happened at that point.

The examples and exercises are also used to shape the interpretation of the mathematical layer. There are many problems in the exercises section asking for proofs that require using not only the definition of derivative but also the theorems about the differentiability of a function, continuity and various others. Those problems can be complex and require the students to be able to work with the theoretical block in applying concepts and providing proofs. This tells the readers that they should focus on theoretical understanding. There are also some computational problems though they mostly involve the chain rule where the techniques of the praxeologies presented previously are of second importance. However, a few exercises do require using the techniques presented. For instance, one problem asks to find the derivative of the functions $\tan(x)$, $cotan(x)$, $\sec(x)$ and $cosec(x)$, by writing them in terms of

$\sin(x)$ and $\cos(x)$ and applying the quotient rule for derivatives. By not presenting any exercises requiring simply a straightforward application of a technique, the author implies the level of expertise the readers should aim at. Another problem where Book C dictates the level of proficiency the readers should attain is in a computational problem requiring the use of the chain rule. In this exercise where eighteen trigonometric functions are asked to be differentiated, the author mentions that it should take the readers no more than twenty minutes to accomplish this task: "It took the author 20 minutes to compute the derivatives for the answer section and it should not take you much longer. Although rapid calculation is not the goal of mathematics, if you hope to treat theoretical applications of the Chain Rule with aplomb, these concrete applications should be child's play..." (Spivak, 1980, p.167). The author doesn't provide many examples and when he does, it is usually to introduce new concepts. For instance, the examples he provided to illustrate the technique to differentiate the product of two functions involved trigonometric functions. He took this opportunity to tell the students how to differentiate such functions which will be studied in length later on in the book. Furthermore, instead of providing an example for each of the following type of tasks: differentiating a constant function, differentiating a power function, differentiating a constant times a function and differentiating the sum or difference of functions, he puts all those tasks together into one abstract example where $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ is differentiated using the techniques. By not providing many examples, the author doesn't give the readers a template to guide them in solving the exercises. Therefore, the readers have to think and figure out how to solve different types of exercises by themselves. This again illustrates the level of understanding the readers are expected to reach and shows limits on how the text can be used as there are no examples whose solutions could be copied to solve the exercises.

Discussion

The author doesn't format the usage of the text as formally as the authors of the other two books do. The low number of examples, however, indicates that the exercises have to be solved reflecting on the techniques, applying them, and referring to technologies and theories; there are no examples to be used as templates to solve the exercises. The author does seem to attempt to format the interpretation of the mathematical layer. He provides comments about the justification of techniques. The author asks the readers to notice the usage of a theorem from a previous chapter that is not part of the reference theory for this technology, as shown above where a theorem and definition for continuity were referenced in the justification of the technique to differentiate $f(x) = 1/g(x)$. The author also lets the readers know the level of proficiency they should attain in the exercises section of the textbook. This section emphasizes problems involving proofs thus telling the readers which part of the mathematical layer they should focus on, mainly the theoretical aspect where the technical aspect, though important, merely supports theoretical applications.

5.4 Discussion

The didactic layers of Books A and B are similar while being quite different from the one of Book C. First of all, both Book A and Book B have a section at the beginning of the book dedicated to telling the readers how to use the text. Book C doesn't have such a section. All three books attempt to format how the text will be interpreted. One way they try to accomplish this is by making sure the steps in examples and justifications of the techniques are well understood thus avoiding misinterpretations. However, they do so at different levels. Book B explains the algebraic manipulations and references the use of previously learnt differentiation rules. Book A merely references the use of limit laws when they are used in the justification of the application of a technique to a particular example. The author of Book C, however, doesn't reference the limit laws or the definitions of derivative and limit; when he has to use continuity in a justification, for example, he lets the readers know what concepts he used. Book A also tries to avoid misinterpretations by alerting the readers of supposedly common mistakes using a special symbol and by explaining what those mistakes are.

The examples and exercises of the textbooks can also let the readers know what part of the mathematical layer they should focus on. Books A and B both seem to convey the importance of mastering techniques as well as an understanding of the different interpretations of the derivative. However, Book B focuses on the graphical interpretation while Book A barely mentions such interpretation instead focusing on the tangent line to a graph and on the rate of change. On the other hand, Book C has very few examples and the exercises focus on a theoretical understanding of a derivative. When computational exercises are presented in Book C, they usually involve the chain rule and the techniques presented are of second importance. Books B and C mostly present exercises that cannot be solved using the worked examples as templates. This is so because in Book C very few examples are provided at all and in Book B most of the exercises require the readers to explain or graph. On the other hand, most worked examples in Book A can be used as templates for most exercises.

6. Conclusions and Discussion

Throughout this paper, I analyzed the mathematical and didactic layers of three textbooks using different frameworks. First, I analyzed the mathematical and didactic layers from the perspective of the Anthropological Theory of the Didactic. In particular, I considered the notions of "praxeology" and "didactic moments" to analyze the mathematical and didactic layers respectively. Next, to further understand the didactic layer, I analyzed the model readers of the textbooks. Finally, I took into account the notion of "formatting the students' behavior" to characterize the possible ways in which the didactic layer may format the readers' relationship with mathematics knowledge.

6.1 Mathematical and Didactic Layers in the Textbooks

To study the mathematical organizations of the three textbooks, I compared the books' praxeologies with a reference praxeology. The results of my analysis showed that the main differences among the three textbooks occurred at the level of the technological and theoretical discourses, that is, in the justifications of the techniques and technologies. Book B has both an incomplete technology and an incomplete theory according to the reference praxeology. Nevertheless, this book still presents the same techniques as the other two books that do have complete (with respect to the reference praxeology) technologies and theories.

A very important missing piece in Books A and B is a unifying praxeology. None of these books provide a regional or local praxeology thus presenting only a *list* of punctual praxeologies. Books C does integrates knowledge into a regional praxeology in relation to the concept of differentiation; the author states that the differentiation techniques wouldn't always be effective and the definition of derivative would sometimes be the only possible approach – thus integrating the punctual praxeologies into the regional praxeology whose defining task is to find the derivative of a function. This is very important as readers might otherwise infer that the punctual praxeologies present all the types of tasks related to differentiating functions and all the techniques available to differentiate any given function (applying one of the differentiation techniques of those punctual praxeologies).

To analyze the didactic layer, I started by looking at the didactic moments. The lack of a unifying regional or local praxeology in Books A and B doesn't support the presence of the didactic moment #3 where the theoretical block is constructed. Regional praxeologies regroup local praxeologies that share the same theory while local praxeologies regroup punctual praxeologies with the same technology. This absence of a unifying praxeology results in a theoretical block that is mostly located in other sections of the textbooks and not explicitly or clearly linked to the punctual praxeologies. Indeed, moment #3 is mainly missing from the sections where know-how blocks of those praxeologies are presented in the textbooks. Moment #3 mainly takes place in previous chapters of the books and it is rarely referenced when the techniques are presented. This results in practico-technical blocks that are disconnected from technologico-theoretical blocks – definitions, theorems, etc., take place in different chapters of the textbooks; furthermore, exercises at the end of the sections seem to reinforce this disconnection. Lithner (2004) and Barbé et al. (2005) found the same type of disconnection between know-how blocks and the knowledge-blocks one investigating reasoning in Calculus textbooks and the implementation of the calculus courses' syllabus in Spanish high-schools, respectively.

Locating the didactic moments, however, was not always possible. Books A and B ease into the presentation of the techniques usually by exploring a task and then elaborating the technique. This is in contrast with Book C that doesn't present an exploration of the tasks and an elaboration of the techniques for four out of six studied praxeologies. Also, Book C deals with the work on the technique (moment #4) in a different way than the other two books. While Books A and B work on the technique in the worked examples, Book C leaves that moment for the exercises section of the textbook where the readers have to complete this moment by themselves.

The analysis of the model readers showed three different intended and implied readers for the textbooks. The author of Book C is writing for an experienced reader who will be able to appreciate the rigor with which he presents the mathematical topics. On the other hand, Book B doesn't require a strong mathematical background from its reader. Book A stands somewhat in-between the two where the author writes for a reader that has completely acquired previously taught topics. An interesting point to rise is that according to Weinberg and Wiesner (2010), the amount of worked examples in Book A and B suggests that the authors are writing for a reader who needs guidance in understanding the topics. However, while those examples do show how to correctly apply the techniques, they don't contribute to a theoretical understanding. The absence of any reference to the theory makes those examples simple routine tasks. Another point that was brought up in the analysis of the model reader is regarding the implied reader and the fact that Books A and B do not provide the definition of tangent line. Since the tangent line to a graph is a crucial part of understanding the derivative, not defining this concept might indicate a major gap in those textbooks. The possible obstacles and misinterpretations that this might lead to are either ignored or neglected by the authors. Book C presents an extensive definition of the concept of tangent line; this seems to be in line with the author's declared intention of developing mathematical rigor in his readers. The authors of Books A and B might have believed that an informal understanding is all that is needed to understand the concepts. The authors would therefore assume that such an informal understanding would have been acquired in the past and therefore no further definition is required. However, as a result those two books are not as self-contained as they seem to attempt to be.

The model reader is made up of the intended and implied reader as well as the empirical reader. The empirical reader is the actual person reading the text. While the empirical reader cannot be determined solely from studying the text, I can still discuss some hypothesis about the empirical reader – in the particular case where the reader is a student. For Books A and B, the empirical student-reader would be similar. First of all, this reader would be a student of calculus I encountering calculus for the first time. The student would be looking to develop his or her technical and conceptual understanding of calculus. However, there is a very important distinction to be made. The observed lack of theory in Book B's praxeologies would make its empirical reader someone who won't attempt to go much further into his or her study of mathematics. Indeed, some crucial theory such as the limit laws or the $\varepsilon - \delta$ definition of limit are missing which would make future studies in calculus and mathematics in general difficult for that reader. On the other hand, the empirical student-reader for Book C won't be the usual student in his or her first attempt at learning calculus. In this case, the empirical reader will most likely be someone who is interested in getting a more rigorous understanding of calculus and uses this book to improve on the calculus already learned. I hypothesize that in this case, the empirical reader will resemble the intended reader.

6.2 Formatting of the Reader's Behavior

I took a closer look at the didactic layer by examining how the books attempt to shape how the readers interpret and use the mathematical layer. All three books do attempt to format how the readers interpret and use the text; they do so at different levels and this is strongly related to the intended reader of each book. For instance, Book B attempts to ensure that most algebraic manipulations are well understood. On the other hand, Books A and C don't have to correct any possible misinterpretations of that sort since their intended readers have a good enough background in algebra – Book B's intended reader may have a weak knowledge of algebra (see § 4.2). Book A also attempts to correct misinterpretations by anticipating what those might be. Indeed, the book warns the readers of expected common mistakes by alerting them using a special symbol and then telling them what that mistake is and why it is an error. However, this might not have the desired effect as some readers may not have thought about the topic in that way and they now have to think about some erroneous interpretation that wasn't on their mind to begin with. Also, in another attempt to format the mathematical layer, Book A's author will present an alternate method to differentiate a product of functions by multiplying out the functions first then differentiating. Then he states that this alternate method is sometimes *easier* to apply, though it is not always possible to do so, than the product rule thus showing a preference for that method when it can be used. It seems that the author misses an opportunity to discuss *easiness* in this context. Furthermore, in the exercises section he asks the reader to differentiate some functions using both methods: applying the product or quotient rule or simplifying first then differentiating, even once asking the readers which method they prefer. The readers are not asked to develop their answers as to why one method is better than the other or in which case a method could be preferred over the other.

The authors, through the examples and exercises, will also let the readers know which "part" of the mathematics they should concentrate on. As the author of Book C writes for an intended reader who is able to appreciate rigor and precision, it naturally follows that the focus is on a theoretical understanding of the concepts. Conversely, Book B focuses on a graphical interpretation of the derivative. Therefore, the authors try to ensure that the possible weak algebraic background of the intended reader is not an obstacle to the understanding of the calculus concepts they want to present. The author of Book A wants to develop a technical competence in its intended reader and the exercises and examples reinforce the importance of this skill. In addition to this aspect of the derivative – the technical skills –, Book A focuses on the interpretations of the derivative as the slope of a tangent line to a graph and as a rate of change. This is in line with the goal of the author of presenting the topics algebraically, numerically and geometrically. In short, depending on which book a person reads, different competences will be developed and he or she might construct different praxeologies related to the topic of differentiation.

6.3 Final Remarks

This project has explored many aspects of the didactic and mathematical layers of the textbooks. However, much more can still be said and pursuing this investigation could help shading light on students' and teachers' uses of mathematical textbooks. For example, in Sierpinska (1997), other analysis tools are provided to further analyze textbooks. One such tool looks at how textbooks integrate the possibility of erroneous interpretations and qualify textbooks as either closed or open. A *closed* text "constructs an implied reader that consists of a unique set of behaviors, codes, and competencies; to match this implied reader, the empirical reader needs to construct specific hypothesis and arrive at specific conclusions based on these hypotheses." (Weinberg and Wiesner, 2010, p. 57) On the other hand, an *open* text "can be meaningfully interpreted with a wide range of behaviors, codes, and competencies and affords multiple valid interpretations." (Eco, 1979; cited in Weinberg and Wiesner, 2010, p. 57) Moreover, an analysis could be performed to further determine how the authors of the textbooks format the utilization of the text by classifying them as either *liberal* or *apodictic*. A textbook can be *liberal* in the sense that the readers have a lot of liberty in the way they choose to use the text. The opposite would be the *apodictic* textbook which offers no margin as to how it's to be used. The readers of apodictic textbooks would have to follow the book page by page and complete every single exercise and activity in order to acquire the concepts presented in the book. Analyzing textbooks from different theoretical perspectives would contribute to our understanding of their possible uses (by teachers and students) in classrooms, and as tools for individual learning or class preparation, for example.

The next logical step following this analysis would be to do an empirical experiment to test the theoretical results obtained here. For instance, it would be interesting to see how the readers really respond to the didactic layer or how the learning is impacted by the absence of certain didactic moments. Another interesting aspect that could be explored is the extent to which the mathematical praxeologies of the textbooks impact the teachers' didactic in the classroom. The different praxeologies could each impose different restrictions and constraints as well as provide opportunities for the teacher. (Barbé et al., (2005) did an experiment of this type, from the perspective of ATD, taking into account the praxeologies as they are defined by syllabi and textbooks in Spanish high schools.) Such empirical studies could shed light on students' and teachers' uses of Calculus textbooks in our classrooms.

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